1.5 Inverse functions

Key question: given a function f, can I find a function g such that $f \circ g = g \circ f = I$.

Ex. $f(x) = 3x$ and $g(x) = \frac{1}{3}x$. Compute $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) = f(g(x))$$
$$= f\left(\tfrac{1}{3}x\right) = 3 \cdot \tfrac{1}{3}x = x$$

$$(g \circ f)(x) = g(f(x)) = g(3x)$$
$$= \tfrac{1}{3} \cdot 3 \cdot x = x$$

We would say that g is the **inverse** of f and use the notation, $g = f^{-1}$.

More questions:

(1) Does every function have an inverse?

(2) How do we compute a function's inverse?

(3) Can we extrapolate properties of f^{-1} from properties of f?

A function f is called **one-to-one** if for every y in the range of f there is exactly one x such that $f(x)=y$.

Ex. $f(x)=x^2$ is not one-to-one on the set of all real numbers.

$f(4) = f(-4) = 16$

two possible x's   one y

However f is one-to-one if the domain is all non-negative numbers.

(!) The inverse function only exists if the original function is one-to-one.

Ex. $f(x)=2x+3$

(*) this function is one-to-one

$f(x)=y$

$2x+3=y$ (*) solve for x

$2x=y-3$

$x=\frac{1}{2}y-\frac{3}{2}=f^{-1}(y)$

(2) **To find the inverse of $f(x)$:**

(A) Let $f(x) = y$.

(B) Solve the above for x.

(C) $x = f^{-1}(y)$

Ex. Compute the inverse of $f(x) = \frac{1}{x-1}$ assuming the domain of f is $(-\infty, 1) \cup (1, \infty)$.

$$f(x) = y$$

$$\frac{1}{x-1} = y$$

$$1 = y(x-1)$$

$$\frac{1}{y} = x - 1$$

$$\frac{1}{y} + 1 = x \quad \Rightarrow \quad f^{-1}(y) = \frac{1}{y} + 1.$$

(3) Domain of f^{-1} = range of f

Range f^{-1} = domain of f.

Ex. Let $f(x) = x^2 + 1$ on the domain $[0,3]$.

(1) What is the range of f.

f is increasing, $f(0)=1$, and $f(3)=10$ therefore the range of f is $[1,10]$

(2) Compute f^{-1}

$y = x^2 + 1$
$y - 1 = x^2$
$\sqrt{y-1} = x \Rightarrow f^{-1}(y) = \sqrt{y-1}$

(3) What is the domain of f^{-1}

$[1,10]$ (the range of f)

(4) What is the range of f^{-1}

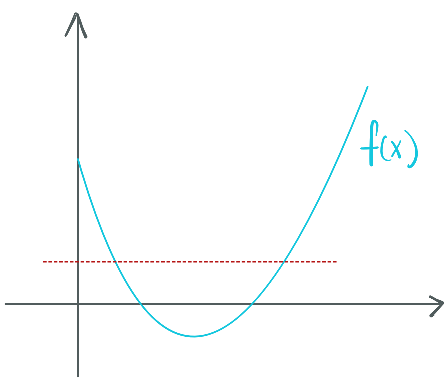
$[0,3]$ (the domain of f)

1.6 Graphical approach to inverse functions

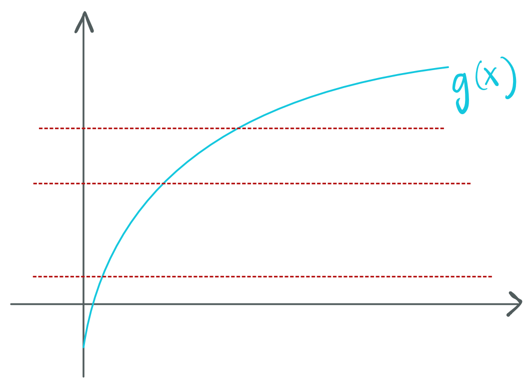
(1) How to tell if a function is one-to-one from it's graph?

Horizontal line test. A function is one-to-one if and only if every horizontal line intersects the function at most once.

Ex. Are the folling functions one-to-one?



No!



Yes!

(2) How to draw a graph of a functions inverse given a graph of the function?

Claim. if the point (a,b) is on the graph of f, then (b,a) is on the graph of

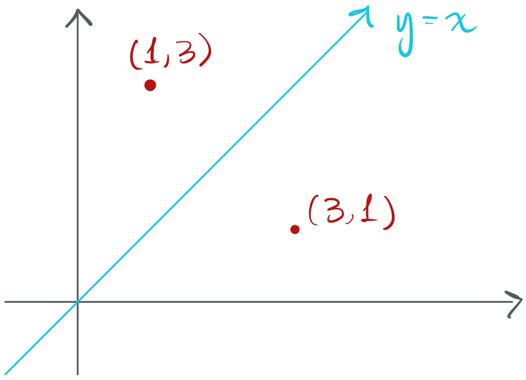
(a,b) on the graph of f

$\Leftrightarrow$ $f(a) = b$

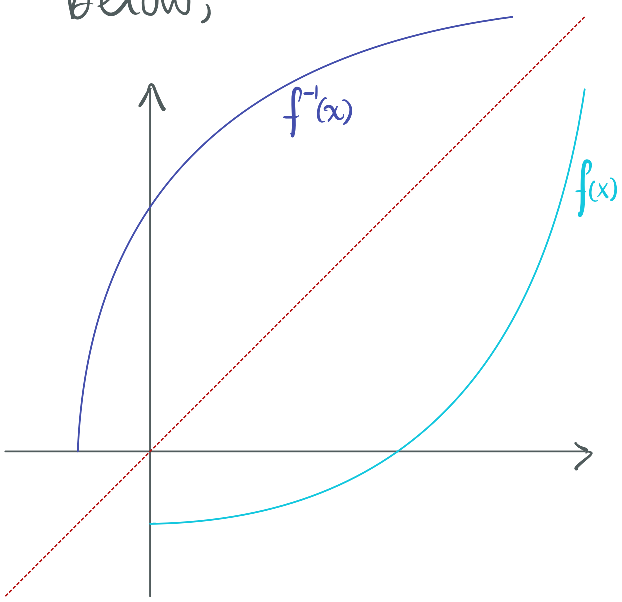
$\Leftrightarrow$ $f^{-1}(b) = a$

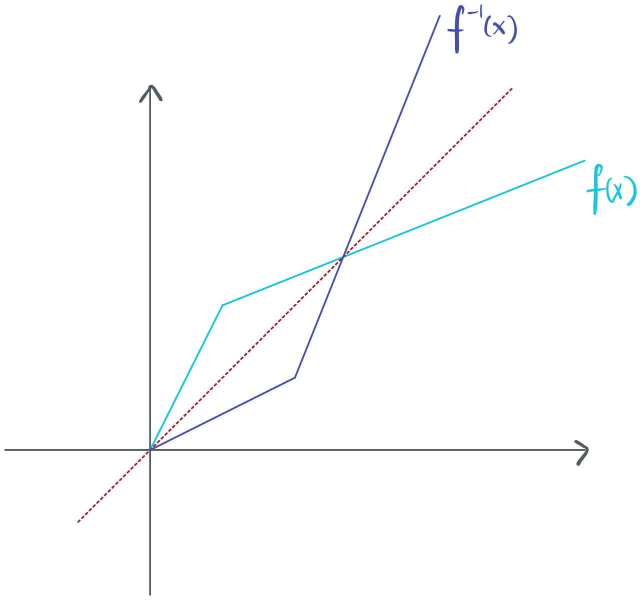
$\Leftrightarrow$ (b,a) is on the graph of (b,a)

To obtain the graph of f^{-1}, flip the graph of f across the line $y=x$.



Ex. Draw the graph of f^{-1} given the graph of f below,





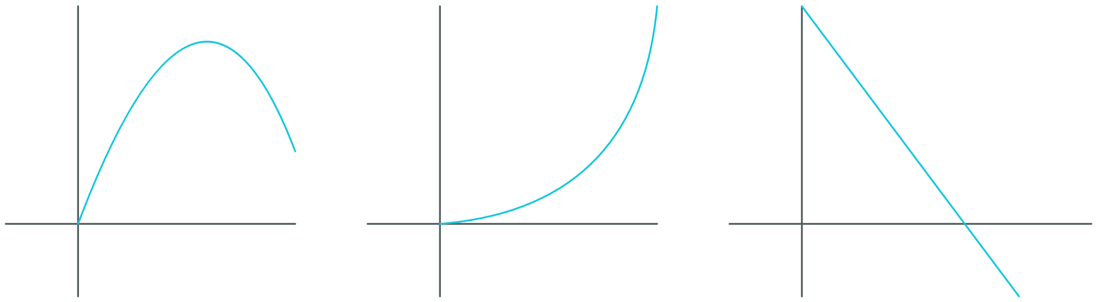
Increasing & decreasing functions.

A function f is called **increasing on an interval** if $f(a) < f(b)$ when $a < b$ for a & b in the interval.
Similarly, f is **decreasing on an interval** if $f(a) > f(b)$ when $a < b$ for a & b in the interval.

Ex. $f(x) = x^2$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

A function is called **increasing (decreasing)** if it is increasing (decreasing) on its entire domain.

Ex.



neither   increasing   decreasing

Every increasing or decreasing function is one-to-one.

And thus invertible!

Inverse of an increasing (decreasing) function is increasing (decreasing).