(3) Let A be an $m \times n$ matrix that represents a linear transformation, $T : \mathbb{R}^n \to \mathbb{R}^m$. Restate the following properties in terms of: rank A, col A, row A, and null A (you don't have to use all of them).

(a) T is onto.

$rank(A) = m$

(b) T is one-to-one.

$\dim Nul(A) = 0$

(c) A is invertible.

$rank(A) = n = m$

(d) T is the zero transformation.

$rank(A) = 0$

(4) Answer the following true or false questions. Justify your answers.

(a) If A is a 6×7 matrix and $RREF(A)$ has 6 pivotsm, then the map given by multiplication by A is one-to-one.

$REF(A)$ has 6 pivots $\Leftrightarrow rank(A) = 6$

by rank thm, $6 + \dim Nul(A) = 7 \Rightarrow \dim Nul(A) = 1$

NOT one-to-one $\Rightarrow$ False.

(b) If A is a 4×5 matrix and B is a 5×3 matrix then rank $A \leq$ rank B.

False,

counter-example: let $B = 0$ and $A \neq 0$.

(c) If A is an $n \times n$ rank matrix such that rank $(A^2) < n$, then rank $A < n$ as well.

True. We can equivalently prove $rank A = n \Rightarrow rank A^2 = n$.

$A^2 : \mathbb{R}^n \xrightarrow{A} \mathbb{R}^n \xrightarrow{A} \mathbb{R}^n \Rightarrow A^2$ is bijection

bijection bijection $\Rightarrow rank(A^2) = n$.

(d) If A is a 4×5 matrix and B is a 5×3 matrix, then rank $(AB) \leq$ rank A.

Let $\underline{x} \in Col(AB) \Rightarrow \underline{x} = AB\underline{y}$ for some $\underline{y}$. Define

or

$B\underline{y} = \underline{z}$, so $\underline{x} = A\underline{z} \Rightarrow \underline{x} \in Col(A)$

$\Rightarrow rank(A) \geq rank(AB)$.