**Rank and Nullity** **Name:**

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(1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Define the following,

(a) rank $T$ = dim Col $A$ = dim Row $A$

(b) the rank theorem (sometimes called the rank-nullity theorem) rank $T$ + dim Nul $T = n$

(2) Let $T$ be a linear transformation represented by a matrix $A$ and suppose $T'$ is the linear transformation represented by the matrix $A^T$, where $(A^T)_{ij} = A_{ji}$.

(a) Let $A$ be an $n \times m$ matrix. What is the domain and codomain of $T$ and $T'$?

$T: \mathbb{R}^m \to \mathbb{R}^n$

$T': \mathbb{R}^n \to \mathbb{R}^m$

(b) Prove that the dim col $A^T$ = dim row $A$.

rows of $A$ = columns of $A^T$
(by definition of transpose.)

(c) Prove that rank $T$ = rank $T'$. Hint: use the REF of $A$.

rank $T$ = rank $T'$ $\Leftrightarrow$ rank $A$ = rank $A^T$

$\Leftrightarrow$ dim Col$(A)$ = dim Row$(A)$

$\|$ $\|$

# pivot columns in REF(A) = # pivot rows in REF(A)

(by def of REF(A)).

(d) When is $T'$ the inverse of $T$? Why is this not always the case?

$\Leftrightarrow AA^T = I \Leftrightarrow$ let $a_i$ be the <u>rows</u> of $A$ (think of as row vectors) then $a_i a_j^T = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

row vector column vector.



This is the definition of an <u>orthogonal</u> matrix