Lecture 3-19

Monday, March 19, 2018 9:21 AM

Warm up problems:

(1) Is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ an orthogonal basis of $\mathbb{R}^2$? Why or why not.

$$\underline{u}_1 \cdot \underline{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [-2 \ 3]$$

$$= -2 + 6 = 4$$

$\underline{u}_1$ is not orthogonal to $\underline{u}_2$ so no.

(2) Is $\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$ an orthonormal basis of $\mathbb{R}^3$? Why or why not.

It is easy to check that each basis element has unit length 1.

Orthogonality:

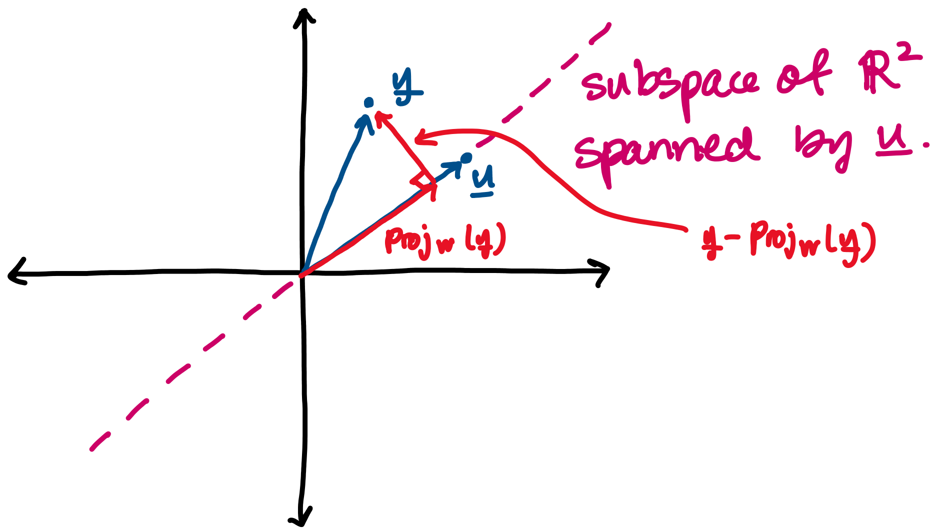
$\underline{u}_1 \cdot \underline{u}_2 = 0 + 2(1) + (-1)2 = 0$

$\underline{u}_1 \cdot \underline{u}_3 = 1(5) + 2(-2) + (-1)(1) = 0$

$\underline{u}_2 \cdot \underline{u}_3 = 0 + 1(-2) + 2(1) = 0$

Yes orthogonal and normal.
Orthogonal vectors are linearly independent so long as non of them are $\underline{0}$. $\Rightarrow$ orthonormal basis of $\mathbb{R}^3$.

Orthogonal Projections



<u>Formalism</u>: let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\underline{u}_1, \ldots, \underline{u}_p\}$, then,

$$\boxed{Proj_W(\underline{y}) = \frac{\underline{y}\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}\underline{u}_1 + \cdots + \frac{\underline{y}\cdot\underline{u}_p}{\underline{u}_p\cdot\underline{u}_p}\underline{u}_p}$$

<u>Examples</u>:

(1) Consider the subspace W spanned by

$$\left\{ \begin{bmatrix} \cdot \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} \cdot \\ 2 \\ -1 \end{bmatrix} \right\}$$

find $\text{Proj}_W(\underline{y})$ where $\underline{y} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

First we should check quickly that this set is orthogonal.

$$\text{Proj}_W(\underline{y}) = \frac{\underline{y} \cdot \underline{u}_1}{\|\underline{u}_1\|^2} \underline{u}_1 + \frac{\underline{y} \cdot \underline{u}_2}{\|\underline{u}_2\|^2} \underline{u}_2$$

$$= \frac{1}{10} 2 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \frac{1}{14} 6 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \frac{1}{7} \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}$$

(2) Let U be the subspace spanned by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and let $y = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Find the minimum distance from y to U.

$\text{Proj}_U(\underline{y})$ is the vector that minimizes this distance, so let's find it.

$$\text{Proj}_U(\underline{y}) = \frac{1}{5}(2(3)+0) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{6}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \hat{y}$$

$$y - \hat{y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 12/5 \\ 6/5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -6/5 \end{bmatrix}$$

$$\|y - \hat{y}\| = \sqrt{\frac{9}{25} + \frac{36}{25}} = \frac{3\sqrt{5}}{5}$$

Question: For a subspace W of $\mathbb{R}^n$, does there exist an orthogonal (orthonormal) basis?

Yes, use Gram-Schmidt.

Examples

Examples:

(1) Use Gram-Schmidt to find an orthogonal basis of W, the subspace spanned by,

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Should check quickly that this set is LI.

1. Set $\underline{v}_1 = \underline{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

basically this is the part of $\underline{x}_2$ parallel to $\underline{v}_1$.

2. Let $\underline{v}_2 = \underline{x}_2 - \text{proj}_{\underline{v}_1}(\underline{x}_2)$

$$\underline{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{5}(6) \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 12/5 \\ 6/5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix}$$