Worksheet Solutions 1-29

Saturday, January 27, 2018 10:11 AM

(1) A set of vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ is linearly independent if $\lambda_1 \underline{v}_1 + \dots + \lambda_n \underline{v}_n = 0$ only if $\lambda_1 = \dots = \lambda_n = 0$.

A set of vectors is linearly dependent if $\lambda_1 \underline{v}_1 + \dots + \lambda_n \underline{v}_n = 0$ where $\lambda_i \neq 0$ for some $i \in 1, \dots, n$.

(2) The linear system is consistent if the <u>last</u> column of the matrix is a <u>free</u> column.

The linear system has a unique solution if only the <u>last</u> column is a <u>free</u> column.

(3) i. consistent, not unique

ii. not consistent or unique

iii. consistent, not unique

iv. consistent and unique

(4) (a) $[\underline{v}_1\ \underline{v}_2 \mid \underline{0}] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right]$

$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right.$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \end{array}\right]$$

since the above system is consistent and unique $\Rightarrow \{\underline{v}_1, \underline{v}_2\}$ is linearly independent.

(b) 3 vectors in $\mathbb{R}^2$

$3 > 2 \Rightarrow \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ linearly dependent

(c) if $\{\underline{v}_1, \underline{v}_2\}$ were linearly dependent there would exist λ such that

$\lambda \underline{v}_1 = \underline{v}_2 \Rightarrow 2\lambda = 0$ and $7\lambda = 1$ and no non-zero number satisfies these

$\Rightarrow \{\underline{v}_1, \underline{v}_2\}$ linearly independent.

(d) $[\underline{v}_1 \; \underline{v}_2 \; \underline{v}_3 \mid \underline{0}] = \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$

drop this column bc it never changes

$$\rightarrow \begin{bmatrix} 1 & 5 & 3 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

System is <u>not</u> unique $\Rightarrow \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are linearly dependent.

(e) $[\underline{v}_1 \ \underline{v}_2 \ \underline{v}_3] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 10 \\ 0 & 5 & 10 \\ 1 & 0 & 3 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & 8 \\ 0 & 5 & 10 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

this column is free

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are linearly dependent

(5) (a) $\begin{bmatrix} 1 & 2 & 0 & 4 \\ -1 & 3 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 5 & 4 & 4 \end{bmatrix}$

$$\to \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 4/5 & 4/5 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 1 & 0 & -8/5 & 12/5 \\ 0 & 1 & 4/5 & 4/5 \end{bmatrix} \\ \begin{array}{cccc} \text{pivot} & \text{pivot} & \text{free} & \text{free} \end{array} \end{array}$$

$$\boxed{\underline{x} = \begin{bmatrix} 12/5 + 8/5\, x^3 \\ 4/5 - 4/5\, x_3 \\ x_3 \end{bmatrix} \qquad \underline{x}_h = x_3 \begin{bmatrix} 8/5 \\ -4/5 \\ 1 \end{bmatrix}}$$

(b)
$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ -1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & -1/2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\boxed{\underline{x} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \qquad \underline{x}_n = \underline{0}}$$

(6) (a) true.

$\underline{b} \in \operatorname{span}\{\underline{a}_1, \dots, \underline{a}_n\} \Rightarrow \underline{b} = \lambda_1 \underline{a}_1 + \dots + \lambda_n \underline{a}_n$ for some $\{\lambda_1, \dots, \lambda_n\}$ not all 0.

$\Rightarrow \underline{b} - \lambda_1 \underline{a}_1 - \dots - \lambda_n \underline{a}_n = 0 \Rightarrow \{\underline{b}, \underline{a}_1, \dots, \underline{a}_n\}$ linearly dependent.

(b) false.

$\underline{a}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\underline{a}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{a}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

two possible solutions (there are ∞-many) are $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\underline{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

(c) false.

Choose any 4 vectors in $\mathbb{R}^3$ and they will be linearly dependent.

i.e.

$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\underline{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(d) false.

Choose $\underline{v}_2$ to be a multiple of $\underline{v}_1$.

i.e.

$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

this geometrically looks like a line.