(1) Define the following terms. Be sure to describe relevant notation used.

Linearly independent:

Linearly dependent:

(2) Consider a linear system represented by the augmented matrix $(\boldsymbol{a_1}\ \boldsymbol{a_2} \cdots \boldsymbol{a_n} \mid \boldsymbol{b})$ and complete the following logical statements:

The linear system is consistent if the ________ column of the matrix is a ________ column.

The linear system has a unique solution if only the ________ column is a ________ column.

(3) Given the following augmented matrices in reduced echelon form determine whether the linear system is consistent and unique.

$$\begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

(4) Determine whether the following set of vectors are linearly independent.

(a) $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(b) $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \boldsymbol{v_3} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(c) $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$, $\boldsymbol{v_3} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

(e) $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} -1 \\ 2 \\ 5 \\ 0 \end{bmatrix}$, $\boldsymbol{v_3} = \begin{bmatrix} 1 \\ 10 \\ 10 \\ 3 \end{bmatrix}$

(5) For the sets of A and $\boldsymbol{b}$ given below solve the linear system $A\boldsymbol{x} = b$ (for $\boldsymbol{x}$). Based on your solution to the nonhomogeneous problem state the solution to the homogeneous problem (without solving it separately!).

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

(6) For the following statements determine whether they are true or false. If false, provide a counter-example to the claim. If true, given brief reasoning (no more than 3 sentences).

(a) If $\boldsymbol{b} \in \text{Span}\{\boldsymbol{a_1}, \ldots, \boldsymbol{a_n}\}$ then the set $\{\boldsymbol{b}, \boldsymbol{a_1}, \ldots, \boldsymbol{a_n}\}$ is linearly dependent.

(b) If $\boldsymbol{b} \in \text{Span}\{\boldsymbol{a_1}, \ldots, \boldsymbol{a_n}\}$ the the solution to $A\boldsymbol{x} = \boldsymbol{b}$ (where $A = [\boldsymbol{a_1} \ \cdots \ \boldsymbol{a_n}]$) is unique.

(c) Let $\boldsymbol{v_1}, \ldots, \boldsymbol{v_m} \in \mathbb{R}^n$. If $m = 4$ and $n = 3$ then the set of vectors is linearly independent.

(d) It $\boldsymbol{v_1}, \boldsymbol{v_2} \in \mathbb{R}^3$, then the $\text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ looks, geometrically, like a plane in $\mathbb{R}^3$.