Lecture 4-4

Wednesday, April 4, 2018 10:09 AM

Singular value decomposition.

For any matrix A we may write,

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

Construction of SVD:

(1) Find orthogonal diagonalization of A^TA

Prove A^TA is symmetric.

$(A^TA)^T = A^TA$.

(2) Set up V and Σ

V: orthonormal eigenvectors

Σ: diagonal are decreasing singular values

(3) Construct U.

First choose $A\underline{v}_1, \dots, A\underline{v}_r$ and then add stuff to get orthonormal basis (if necessary). You may need gram-schmidt.

Ex. Find SVD of $A = \begin{bmatrix} 2 & -2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$$

$$\begin{aligned} \det(A^TA - \lambda I) &= (5-\lambda)(5-\lambda) - 9 \\ &= 25 - 10\lambda + \lambda^2 - 9 \\ &= \lambda^2 - 10\lambda + 16 \\ &= (\lambda-2)(\lambda-8) \end{aligned}$$

$\lambda_1 = 8 \quad \lambda_2 = 2$

$\sigma_1 = 2\sqrt{2} \quad \sigma_2 = \sqrt{2}$

$$A^TA - 8I = \begin{bmatrix} -3 & -3 \\ \end{bmatrix} \Rightarrow \underline{v}_1 = \frac{1}{\,}\begin{bmatrix} 1 \\ \end{bmatrix}$$

$\begin{bmatrix} -3 & -3 \end{bmatrix}$ $\qquad \sqrt{2}\begin{bmatrix} -1 \end{bmatrix}$

$$A^TA - 2I = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \Rightarrow \underline{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

(*) U should be 3×3.

$$\underline{u}_1 = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Need one more, which is obviously

$$\underline{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$