Solutions

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(1) Orthonormal basis of

$$W = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \right\}$$

Use Gram-Schmidt,

$$\underline{u}_1 = \underline{x}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

$$\underline{u}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}_1} \cdot \underline{x}_1 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} - \frac{(-1)}{13} \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16/13 \\ -2 \\ -24/13 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 16 \\ -26 \\ -24 \end{bmatrix}$$

Now we need to normalize $\underline{u}_1$ and $\underline{u}_2$ to get an orthonormal basis.

$$\boxed{\begin{aligned} \underline{v}_1 &= \frac{1}{\|\underline{u}_1\|} \underline{u}_1 = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \\ \underline{v}_2 &= \frac{1}{\|\underline{u}_2\|} \underline{u}_2 = \frac{1}{\sqrt{1508}} \begin{bmatrix} 16 \\ -26 \\ -24 \end{bmatrix} \end{aligned}}$$

(2) Compute a QR factorization of

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A \quad \begin{pmatrix} 2 & 0 & 2 \end{pmatrix}$$

Use Gram-Schmidt on columns of A to find Q.

$$\underline{u}_1 = \underline{x}_1 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$$\underline{u}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}_1} \underline{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{8} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 1 \\ -1/2 \end{pmatrix} \text{ fine to write as } \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\underline{u}_3 = \underline{x}_3 - \frac{\underline{x}_3 \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}_1} \underline{x}_1 - \frac{\underline{x}_3 \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2/3 \\ 2/3 \\ 2/3 \end{pmatrix} \text{ fine to write as } \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$$

Normalize the above vectors,

$$\underline{v}_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \qquad \underline{v}_3 = \frac{1}{2\sqrt{3}} \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\underline{v}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$$

$$\boxed{Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}}$$

$$R = Q^T A = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\boxed{R = \begin{pmatrix} 1 & 1/\sqrt{2} & \sqrt{2} \\ 0 & 3/\sqrt{6} & -2/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix}}$$

(3) (a) Is $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ orthogonal?

No. $\begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \neq 0.$

(3) (b) Will the Gram-Schmidt procedure produce the same orthonormal basis if $\underline{x}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and if $\underline{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

No. Let $\mathcal{B}$ be the basis produced

by starting with $x = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and let $\mathcal{B}'$

by starting with $\underline{x}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and let $\mathcal{B}'$ be the basis produced by starting with $\underline{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

If $\mathcal{B} = \mathcal{B}'$, then $\mathcal{B} \ni \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ but $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not orthogonal to $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$, and thus a contradiction to $\mathcal{B} = \mathcal{B}'$.

Compute $\mathcal{B}$:

$$\underline{u}_1 = \underline{x}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\underline{u}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{x}_1}{\underline{x}_1 \cdot \underline{x}_1} \underline{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Normalize $\underline{u}_1$ and $\underline{u}_2$: $\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$

Compute $\mathcal{B}'$:

$$\underline{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\underline{u}_2 = \underline{x}_1 - \frac{\underline{x}_1 \cdot \underline{x}_2}{\underline{x}_1 \cdot \underline{x}_1} \underline{x}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12/5 \\ -6/5 \end{bmatrix} \sim \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Normalize $\underline{u}_1$ and $\underline{u}_2$: $\boxed{\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}}$

(4) Diagonalize the following if possible.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Since A is lower triangular the eigenvalues are $\lambda = 1, 3, 2$.

Note we know it's diagonalizable because the eigenvalues are distinct.

$$\boxed{D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

Now let's compute the eigenvectors.

$\lambda = 1$:

$$A - I = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$\lambda = 3$:

$$A - 3I = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = 2$:

$$A - 2I = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{P = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}}$$

(5) Let $T: \mathbb{P}_2 \to \mathbb{P}_2$ be a transformation that sends $at^2+bt+c \mapsto ct^2+at+b$.

(a) Show that T is linear.

Let $p_1 = at^2+bt+c$

$p_2 = et^2+ft+g$

$p_1+p_2 = (a+e)t^2+(b+f)t+(c+g)$

$$T(p_1+p_2) = (c+g)t^2+(a+e)t+(b+f)$$
$$= ct^2+at+b+gt^2+et+f$$
$$= T(p_1)+T(p_2).$$
$$T(\alpha p_1) = \alpha ct^2+\alpha at+\alpha b$$
$$= \alpha(ct^2+at+b) = \alpha T(p_1)$$

And so T is a linear transformation.

(5)(b) Consider the standard basis of $\mathbb{P}_2$, $B = \{1, t, t^2\}$, compute the matrix T_B.

$$T_B = \left[[T([1]_B)]_B \quad [T([t]_B)]_B \quad [T([t^2]_B)]_B \right]$$

$$\boxed{T_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}$$

(5)(c) Compute the eigenvalues and eigenvectors of $T_{\mathcal{B}}$.

$$\det(T_{\mathcal{B}} - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda((\lambda^2)) - 1(-1)$$

$$= 1 - \lambda^3$$

There is only one real eigenvalue, the other two are complex (and we won't worry about them).

$\boxed{\underline{\lambda = 1}:}$

$$A - I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad \boxed{\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

In terms of the transformation, this tells us T has a "fixed point" for polynomials of the form,
$p(t) = \lambda t^2 + \lambda t + \lambda$.

(6) Find examples of the following

Find examples of the following

(a) A matrix that is diagonalizable but not invertible

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = A$$

(b) A matrix with rank = 1

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(c) An orthonormal basis of $\mathbb{R}^3$ that's not the standard basis.

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(d) A square matrix with a different row space and column space.

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$