Lecture 2-20

Tuesday, February 20, 2018 10:37 AM

Def. Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_p\}$ is a basis for H if,

(i) $\mathcal{B}$ is linearly independent

(ii) $H = \text{Span}\{\underline{b}_1, \dots, \underline{b}_p\}$

Basis can be thought of as the "smallest set of vectors" that span a given vector space.

Question: Does a vector space V always have a unique basis?

Answer: NO. Consider the two bases of $\mathbb{R}^2$,

$\{\underline{e}_1, \underline{e}_2\}$, $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$.

Question: Let V be a vector space and $\mathcal{B}_1 = \{\underline{b}_1, \dots, \underline{b}_n\}$, $\mathcal{B}_2 = \{\underline{v}_1, \dots, \underline{v}_m\}$ be two bases of V. What can be said about the relationship between m and n?

Answer: $m=n$.

Ex. Given the set of vectors V, determine whether they form a basis of Span V. If not find a basis of Span V.

(1) $V=\left\{\begin{bmatrix}0\\2\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$

Yes, V is a basis of Span V.

Just needed to check that V is linearly independent.

(2) $V=\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}0\\0\\4\end{bmatrix},\begin{bmatrix}2\\4\\4\end{bmatrix}\right\}$

No, a basis of Span V is $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix},\begin{bmatrix}0\\0\\4\end{bmatrix}\right\}$

(3) $V=\left\{\begin{bmatrix}0\\2\\1\end{bmatrix},\begin{bmatrix}0\\4\\0\end{bmatrix}\right\}$

Yes.

Ex. Given the following A, determine a basis for col(A) and nul(A).

(1) $A=\begin{bmatrix}1&2&0&0&2\\0&0&1&0&-1\\0&0&0&1&3\end{bmatrix}$

Basis of Col(A) $=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ ← each pivot column.

Basis of Nul(A) $=\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\\-3\\1\end{bmatrix}\right\}$ ← one for each free column but NOT the free columns

represents $-2a_1+a_2=0$ represents $-2a_1+a_3-3a_4+a_5=0$

(2) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Basis of $Col(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Basis of $Nul(A) = \left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\}$

Def. The dimension of a vector space V, denoted $\dim V$, is the number of vectors in a basis of V.

Notice the pattern: for $n \times m$ matrix A, $\dim(Col(A)) + \dim(Nul(A)) = m$.

Can you prove this fact?

Ex. Given the following matrix A, determine the $\dim(Col(A))$ and $\dim(Nul(A))$

(1) $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

$REF(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \dim(Col(A)) = 2$

$\dim(Nul(A)) = 0$

(2) $A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$

$\dim(\text{Col}(A)) = 3$

$\dim(\text{Nul}(A)) = 1$

Challenge: Let $C^{\infty}(\mathbb{R}) = \{f: \mathbb{R} \to \mathbb{R} : f \text{ is infinitely differentiable } \forall x \in \mathbb{R}\}$, $\dim(C^{\infty}(\mathbb{R})) = \infty$, what might be a basis of $C^{\infty}(\mathbb{R})$?

Hint: think about Taylor Series.