Lecture 1-31

Monday, January 29, 2018 1:11 PM

Let's talk about section 1.7 a little bit more...

Ex. Let $\underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $\underline{v}_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Justify why the set of vectors is linearly dependent and find λ_1, λ_2, and λ_3 such that $\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \lambda_3 \underline{v}_3 = 0$.

3 vectors in $\mathbb{R}^2$ cannot be linearly independent by theorem 8 in section 1.7.

Use augment matrix to find λ_1, λ_2, and λ_3.

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 3 & 6 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\underline{\lambda} = \begin{bmatrix} 2\lambda_3 \\ -2\lambda_3 \\ \lambda_3 \end{bmatrix} \qquad \boxed{2\underline{v}_1 - 2\underline{v}_2 + \underline{v}_3 = 0}$$

Challenging Problem. Prove that $A\underline{x} = \underline{b}$ ① has more than one solution if and only if $A\underline{x}_h = \underline{0}$ ② has more than one solution

Note: "① if and only if ②" mean you must show ① ⇒ ② and ② ⇒ ①.

$A\underline{x}_h = \underline{0}$ has more than one solution ⇔ $A\underline{x}_h = \underline{0}$ has a non-zero solution ⇔ the columns of A are linearly dependent.

② ⇒ ①:

Obvious from the fact that $\underline{x} = \underbrace{\underline{x}_p}_{\text{particular solution}} + \underbrace{\underline{x}_h}_{\text{homogeneous solution}}$

① ⇒ ②:

NOT the same!

Let $\underline{x}_1$ and $\underline{x}_2$ be solutions such that $A\underline{x}_1 = \underline{b}$ and $A\underline{x}_2 = \underline{b}$.

then, $A\underline{x}_1 - A\underline{x}_2 = \underline{b} - \underline{b}$

$A(\underline{x}_1 - \underline{x}_2) = \underline{0}$

$\Rightarrow$ both $\underline{0}$ and $\underline{x}_1 - \underline{x}_2$ satisfy $A\underline{x}_h = 0$.

$\square$

Reference: section 1.8

Topics and important definitions:

- Linear transformation, $T: \mathbb{R}^n \to \mathbb{R}^m$
- Domain, codomain, image and range

codomain $\neq$ range!

Question: when does codomain = range?

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be represented by the matrix $A = [\underline{a}_1 \ \underline{a}_2 \cdots \underline{a}_n]$ (note $\underline{a}_i \in \mathbb{R}^m$ for all i).

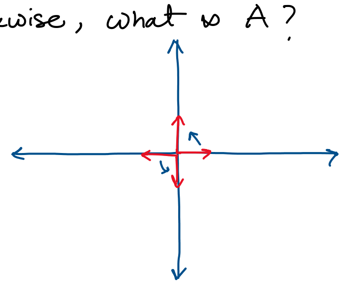
Restated the question is when does the range of T = all of $\mathbb{R}^m$

When,

$$\boxed{\text{Span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\} = \mathbb{R}^m}$$

Fact: $T: \mathbb{R}^n \to \mathbb{R}^m$ is represented by matrix A, then A has n columns and m rows.

Ex. Let $T(\underline{x}) = A\underline{x}$ and suppose $T(\underline{x})$ takes a point in the plane and rotates it $90°$ counter clockwise, what is A?



$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$a_{12} = -1$

$a_{22} = 0$

$$\begin{bmatrix} a_{11} & -1 \\ a_{21} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$a_{11} = 0$

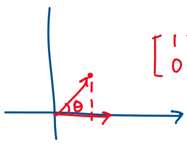
$a_{21} = 1$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

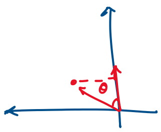
Ex. Show that if $T(\underline{x})$ rotates $\underline{x}$ by θ, then

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Same process as above, but what do the 4 vectors drawn above transform into?



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Challenging Problem. (from last time) Given the definition of linear transformation:

A transformation is linear if:

(i) $T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v}) \quad \forall\, \underline{u}, \underline{v}$

(ii) $T(c\underline{u}) = cT(\underline{u}) \quad \forall c$ and $\forall\, \underline{u}, \underline{v}$

Show that (i) ALMOST implies (ii). What else is necessary?

Hint: Show that all linear transformations are continuous. (You'll need that ϵ-δ definition of continuity though

(i) $\Rightarrow$ (ii) for all $s \in \mathbb{Q}$ (rational numbers). Sketch of proof:

- Show true for any $k \in \mathbb{Z}$ (integer) by noting that $k\underline{x} = \underline{x} + \underline{x} + \cdots + \underline{x}$.
- Show that $T(0) = 0$ (why does this prove it for $s = 0$?)
- Show for $s = \frac{a}{b}$ $a, b \in \mathbb{Z}$ (hint: use $\frac{b}{b} = 1$)

If we assume T is continuous, then (i) $\Rightarrow$ (ii) for all $s \in \mathbb{R}$.

$\square$