Lecture 2-5 and 2-7

Wednesday, January 31, 2018 11:57 AM

Where we left off,

- A linear transformation, $T: \mathbb{R}^n \to \mathbb{R}^m$, can be represented by an $m \times n$ matrix, A, where $T(\underline{x}) = A\underline{x}$.

domain ($\mathbb{R}^n$), codomain ($\mathbb{R}^m$); rows (m), columns (n)

<u>Fact</u>. Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ and let $T(\underline{x}) = A\underline{x}$. The matrix A can be computed by,

$$A = \left[T(\underline{e}_1) \ \cdots \ T(\underline{e}_n) \right]$$

Where $\underline{e}_1, \dots, \underline{e}_n$ are the standard basis vectors in $\mathbb{R}_n$.

<u>Ex 1</u>. Let $T(\underline{x}) = A\underline{x}$ be the map that sends a vector $\underline{x} \in \mathbb{R}^3$ to a corresponding point on the x-y plane $\left(\underline{x} = (x, y, z) \longrightarrow (x, y, 0) \right)$, where the x-y plane is thought of as $\mathbb{R}^2$.

"smash" 3 space to a plane.

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$$\left.\begin{aligned} T(e_1) &= e_1 \\ T(e_2) &= e_2 \\ T(e_3) &= 0 \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Is the above map onto?

Yes, for any $\underline{y} \in \mathbb{R}^2$ we can find an $\underline{x} \in \mathbb{R}^3$ such that $T(\underline{x}) = \underline{y}$ (i.e. if $\underline{y} = (x, y)$ let $\underline{x} = (x, y, 0)$.

Is the above map one-to-one?

No, let $\underline{x}_1 = (x, y, 0)$ and $\underline{x}_2 = (x, y, 1)$, then $T(\underline{x}_1) = T(\underline{x}_2) = (x, y)$.

For the following let $T(\underline{x}) = A\underline{x}$. For the specified A, determine if $T(\underline{x})$ is one-to-one and onto.

<u>Ex 2.</u> $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

(*) Sol'n of $T(\underline{x}) = \underline{0}$ is unique.

So by theorem 11, T is one-to-one. Also, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ span $\mathbb{R}^2 \Rightarrow$ T is onto.

<u>Ex 3.</u> $A = \begin{bmatrix} 2 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

3 vectors, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, in $\mathbb{R}^2$ are not linearly independent $\Rightarrow$ (thm 12) T is not one-to-one.

$$\begin{bmatrix} 2 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

these columns clearly span $\mathbb{R}^2$

$\Rightarrow T$ is onto.

<u>Ex 4</u>. $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

<u>col(1)</u> + <u>col(2)</u> = <u>col(3)</u> $\Rightarrow$ not linearly independent

$\Rightarrow T$ is not one-to-one

Also since we have 3 vectors that are linearly dependent in $\mathbb{R}^3 \Rightarrow$ they don't span $\mathbb{R}^3 \Rightarrow T$ is not onto.

Now onto $\boxed{2.1}$ <u>Matrix Operations</u>.

$$\left\{ \begin{matrix} \text{linear transformations} \\ \text{from } \mathbb{R}^n \text{ to } \mathbb{R}^m \end{matrix} \right\} = \left\{ (m \times n) \text{ Matrices} \right\}$$

Add transformations $\quad = \quad A+B$

$T_A(\underline{x}) + T_B(\underline{x}) = T_{A+B}(\underline{x})$

both $\mathbb{R}^n \to \mathbb{R}^m$ $\qquad$ both $m \times n$

Multiply by a constant $\quad = \quad \lambda A$

$\lambda T_A(\underline{x}) = T_{\lambda A}(\underline{x})$

Composition of transformations $=$ AB $\quad$ new matrix is $s \times n$

$$(T_A \circ T_B)(\underline{x}) = T_{AB}(\underline{x})$$

$\mathbb{R}^m \to \mathbb{R}^s$ $\quad$ $\mathbb{R}^n \to \mathbb{R}^m$ $\quad$ $\mathbb{R}^n \to \mathbb{R}^s$

$(s \times m)$ $(m \times n)$

inner ones agree.

Ex 5. Prove that, in general, $(A^k)_{ij} \neq (A_{ij})^k$.

Give an example where $(A^k)_{ij} = (A_{ij})^k$.

Find a counter example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$\neq 1!$

It is true for any diagonal matrix,

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$$

$$A^k = \begin{bmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{bmatrix}$$

Ex Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & k \\ 0 & -1 \end{bmatrix}$, for what value of k (if any) does $AB = BA$?

$$AB = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & k \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & k-3 \\ 0 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & k \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6+2k \\ 0 & -2 \end{bmatrix}$$

$BA = AB$ when $6+2k = k-3$

$$k = -9$$

Ex. Same as above,

$$A = \begin{bmatrix} 2 & 2 \\ 1 & k \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 0 \\ 3k & k \end{bmatrix} \quad BA = \begin{bmatrix} -1 & -k \\ 7 & 6+k \end{bmatrix}$$

impossible

Ex. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 0 \\ 3k & 3 \end{bmatrix} \quad BA = \begin{bmatrix} 2 & 0 \\ 2k & 3 \end{bmatrix}$$

$$k = 0$$

Recall the augmented matrix form we produced for $A\underline{x} = \underline{b}$. We can use the same form for problems of the type $AB = I$.

Ex Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and suppose $AB = I$. Find B?

the matrix that

appears on this side after elementary row operations is B

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1/3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/3 & -1 & 1 \end{array}\right]$$

B!

What are the limitations of the above?

A & B must be square

in general for A to have an inverse it must be square, but don't confuse this with the fact that $AB = I$ is possible for non-square A and B.

Let $AB = I$. Compute B for the following As,

(1) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(2) $A = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$

13) $A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix}$

14) $A=\begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$