Lecture 3-7

Tuesday, March 6, 2018 10:55 AM

Last time: - eigenvalues
- characteristic poly
- eigenvectors
- eigenspace.

Today: - diagonalization

An $n \times n$ matrix is diagonalizable iff it has n distinct eigenvectors

Note: eigenvectors are always LI.

$\Updownarrow$

For all eigenvalues of, the algebraic multiplicity must equal the dim. of the eigenspace.

Fact. if $n \times n$ matrix A has n distinct eigenvalues A is diagonalizable.
(*) doesn't mean A has to have n distinct eigenvalues to be diagonalizable.

Ex Are the following diagonalizable?

(1) A is 2×2 with eigenvalues $\lambda = 3, 0$ yes.

(2) A is 3×3 with eigenvalues $\lambda = 3, 2$ and $\dim(E_3) = 2$. yes.

(3) A is 4×4 with 3 eigenvalues

and dim of all eigenspaces is 1.

No.

Ex For the following, compute D, P, and $A = PDP^{-1}$.

(1) $\lambda = 3 \rightarrow \underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2 \rightarrow \underline{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix}$$

Check: $\begin{bmatrix} 5/2 & 1/2 \\ 1/2 & 5/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$p(\lambda) = (3-\lambda)(2-\lambda)$

(2) $\lambda = 1 \rightarrow \underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\underline{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 0 \rightarrow \underline{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\quad P=\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\quad P^{-1}=\frac{1}{2}\begin{bmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{bmatrix}$$

$$A=PDP^{-1}$$

$$=\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&1&-1\\-1&1&1\\1&-1&1\end{bmatrix}$$

Important notes:

- Order in D & P matter.
- Diagonalizable $\neq$ invertible
  - (*) $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ is invertible but not diagonalizable
  - (*) $\begin{bmatrix}0&0\\0&1\end{bmatrix}$ is diagonalizable but not invertible.

Ex. Determine whether A is diagonalizable, if so compute D and P.

(1) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda = 2, 1$

↑ alg. mult. 2

$\underline{\lambda = 2}$:

$A - 2I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Nullspace only has dim 1 and so the eigenspace has dim 1

$\Rightarrow$ A is <u>NOT</u> diagonalizable.

(2) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2$$

$$= 4 - 5\lambda + \lambda^2 + 2$$

$$= (\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 3, 2.$$

two distinct eigenvalues

$\Rightarrow$ diagonalizable.

$\underline{\lambda = 3}$:

$A - 3I = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}$

$\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\underline{\lambda = 2}$:

$A - 2I = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$$\underline{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$