Lecture 2-28

Wednesday, February 28, 2018 9:30 AM

Let $\underline{x} \in \mathbb{R}^n$, $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

We know in the standard basis,

$$\mathcal{E} = \{ \underline{e}_1, \ldots, \underline{e}_n \}$$

We can write $\underline{x}$ as a linear combination,

$$\underline{x} = x_1 \underline{e}_1 + \cdots + x_n \underline{e}_n$$

Thus we say, $[\underline{x}]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{x}$.

Given an alternative basis $\mathcal{B} = \{ \underline{b}_1, \ldots, \underline{b}_n \}$ of $\mathbb{R}^n$ we can write $\underline{x}$ as a linear combination of these vectors

$$\underline{x} = \lambda_1 \underline{b}_1 + \cdots + \lambda_n \underline{b}_n$$

And so we say, $[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$. Note that $[\underline{x}]_{\mathcal{B}} = \underline{x}$ only if $\mathcal{B}$ is the standard basis.

Ex. Given $\mathcal{B} = \{ \underline{b}_1, \ldots, \underline{b}_n \}$ and $\underline{x} = \lambda_1 \underline{b}_1 + \cdots + \lambda_n \underline{b}_n$, determine $[\underline{x}]_{\mathcal{B}}$ and $\underline{x}$.

(1) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, $\underline{x} = 3\underline{b}_1 - 3\underline{b}_2$

(2) $\mathcal{B} = \{ [2 \ldots], [1 \ldots] \}$

(2) $\mathcal{B} = \left\{ \begin{bmatrix} \cdot \\ 3 \end{bmatrix}, \begin{bmatrix} \cdot \\ 0 \end{bmatrix} \right\}$, $\underline{x} = 2\underline{b}_1 + \underline{b}_2$

(3) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, $\underline{x} = 2\underline{b}_1 + \underline{b}_2 - 3\underline{b}_3$

Now let's say we have a vector $\underline{x}$ and we want to know how to represent it in a basis $\mathcal{B}$, that is find $[\underline{x}]_{\mathcal{B}}$.

Solve one of the two:

(a) $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} \underline{x} = [\underline{x}]_{\mathcal{B}}$ — matrix is harder to find, straight-forward multiplication

(b) $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} [\underline{x}]_{\mathcal{B}} = \underline{x}$ — matrix is easy to find, solve augmented system

<u>Ex</u>.

(1) Let $\underline{x} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

compute $[\underline{x}]_{\mathcal{B}}$ using (b).

$$\underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \left[[\underline{b}_1]_{\mathcal{E}} \ [\underline{b}_2]_{\mathcal{E}} \right] = \left[\underline{b}_1 \ \underline{b}_2 \right]$$

(2) Let $\underline{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$ compute $[\underline{x}]_{\mathcal{B}}$ using (a).

$$\underset{\mathcal{B}\leftarrow\mathcal{E}}{P} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}^{-1}$$

Changing between two bases that aren't standard.

Consider two bases, $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ and $\mathcal{C} = \{\underline{c}_1, \dots, \underline{c}_n\}$ in $\mathbb{R}^n$

$$[\,\underline{c}_1 \cdots \underline{c}_n \mid \underline{b}_1 \cdots \underline{b}_n\,] \sim [\, I \mid [\underline{b}_1]_{\mathcal{C}} \cdots [\underline{b}_n]_{\mathcal{C}}\,]$$
$$= [\, I \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,]$$

$$[\,\underline{b}_1 \cdots \underline{b}_n \mid \underline{c}_1 \cdots \underline{c}_n\,] \sim [\, I \mid [\underline{c}_1]_{\mathcal{B}} \cdots [\underline{c}_n]_{\mathcal{B}}\,]$$
$$= [\, I \mid \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\,]$$

Ex. let $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

Suppose $[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ compute $[\underline{x}]_{\mathcal{C}}$