Lecture 3-12

Friday, March 9, 2018 5:15 PM

Matrix of an abstract linear transformation.

For a transformation, $T: \mathbb{R}^n \to \mathbb{R}^m$, there is a matrix representation A (an $m \times n$ matrix).

We can think of a typical linear transformation as,

$T: (\mathcal{E}, \mathbb{R}^n) \to (\mathcal{E}, \mathbb{R}^m)$ (Represented by matrix A).

standard basis.

What if we wanted to understand $T: (\mathcal{B}, \mathbb{R}^n) \to (\mathcal{C}, \mathbb{R}^m)$?

An important diagram:

$$(\mathcal{E}, \mathbb{R}^n) \xrightarrow{A} (\mathcal{E}, \mathbb{R}^m)$$

$$\begin{array}{ccc} \Big\vert \; \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} & & \Big\downarrow \; \underset{\mathcal{C}\leftarrow\mathcal{E}}{P} \\ (\mathcal{B}, \mathbb{R}^n) & \xrightarrow{A_{\mathcal{B},\mathcal{C}}} & (\mathcal{C}, \mathbb{R}^m) \end{array}$$

From the above we see,

$$\boxed{A_{\mathcal{B},\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P} \, A \, \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}}$$

<u>Ex</u> Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ represented by

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

And we want to understand $A_{\mathcal{B},\mathcal{C}}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \qquad \underset{\mathcal{E}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\underset{C\leftarrow \mathcal{E}}{P} = \underset{\mathcal{E}\leftarrow C}{P^{-1}} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A_{B,C} = \underset{C\leftarrow \mathcal{E}}{P}\, A \underset{\mathcal{E}\leftarrow B}{P}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 0 & 0 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

Ex. Find a basis C such that $A_{\mathcal{E},C} = \mathbb{1}$ given

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Since $\underset{\mathcal{E}\leftarrow \mathcal{E}}{P} = \mathbb{1}$ just need to find $\underset{C\leftarrow \mathcal{E}}{P}$ s.t.

$$\underset{C\leftarrow \mathcal{E}}{P} A = \mathbb{1} \Rightarrow \underset{C\leftarrow \mathcal{E}}{P} = A^{-1}$$

recall

$P_{\varepsilon \leftarrow C} = [\, \underline{c}_1 \;\cdots\; \underline{c}_n \,]$

$\Rightarrow P_{\varepsilon \leftarrow C} = A$

$$\Rightarrow C = \left\{ \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

__Ex__. Now find a basis $\mathcal{B}$ such that $A_{\mathcal{B},\varepsilon} = \mathbb{1}$,

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

$$A \, P_{\varepsilon \leftarrow \mathcal{B}} = \mathbb{1} \Rightarrow P_{\varepsilon \leftarrow \mathcal{B}} = A^{-1}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} = [\, \underline{b}_1 \;\; \underline{b}_2 \,]$$

$$\mathcal{B} = \left\{ \frac{1}{2} \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

Now we wish to know if, for some A, there is a given basis $\mathcal{B}$ such that $A_{\mathcal{B},\mathcal{B}} = D$.

diagonal matrix, think

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

The answer is yes <u>if</u> A is diagonalizable.

The appropriate basis is the one of eigenvectors of A.