Q1.1 False.

This can fail if eigenvectors come from the same eigenspace.

Here's a counter example...

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A is symmetric

Both eigenvectors, $Av_1 = 2v_1$
$Av_2 = 2v_2$

$v_1 \cdot v_2 = 1 \neq 0$.

Q1.2 True.

If A is a 2×2 orthogonal matrix it is either,

$$\begin{bmatrix} \pm1 & 0 \\ 0 & \pm1 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Diagonal

Rotation $\downarrow$ <u>NOT</u> diagonalizable over $\mathbb{R}$, det $= 1$.

Reflection $\downarrow$ diagonalizable over $\mathbb{R}$, det $= -1$.

Q1.3 False.

They have the same non-zero singular values.

Let's exhibit a counter example,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

singular values of A are 1, 1.

Singular value of A^T are 1, 1, 0.

Q1.4 False.

Choose something not positive semi-definite,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \qquad \lambda = -2, 4$$

$$\sigma = 2, 4 \leftarrow \text{always positive!}$$

Q2 Let's go through the statements,

- the columns must also be normalized
- orthogonal $\not\Rightarrow$ diagonalizable over $\mathbb{R}$ $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$
- TRUE! $A+A^T$ is symmetric $\Rightarrow$ orthogonally diagonalizable.
- only true if A is orthogonally diagonalizable.

Q3.1 False.

Let's plug y_1+y_2 into the DE,

$$(y_1+y_2)'' + 2(y_1+y_2) + 2(y_1+y_2) \overset{?}{=} e^t$$

$$\underbrace{y_1'' + 2y_1' + 2y_1}_{=e^t} + \underbrace{y_2'' + 2y_2' + 2y_2}_{=e^t} \overset{?}{=} e^t$$

$$2e^t \neq e^t$$

Q3.2 True

A basis for the space of solutions is $\{e^{-t}, te^{-t}\}$.

Q3.3 True

By uniqueness of IVP.

in this case we have
2 unknowns and 2 equations

Q3.4 False.

Just plug it in,

$$(ce^t)'' - ce^t \stackrel{?}{=} e^t$$

$$ce^t - ce^t \neq e^t$$

Q4 Lets show the second choice is correct by checking the solution in the DE,

$$y = e^t \sin(t + \pi/4)$$

$$y' = e^t \sin(t + \pi/4) + e^t \cos(t + \pi/4)$$

$$y'' = 2e^t \cos(t + \pi/4)$$

$$2e^t \cancel{\cos(t+\pi/4)} - 2e^t \cancel{\sin(t+\pi/4)} - 2e^t \cancel{\cos(t+\pi/4)} + 2e^t \cancel{\sin(t+\pi/4)} = 0$$

Q5. $r^2 - 4r - 5 = 0$

$$(r-5)(r+1) = 0 \Rightarrow r = 5, -1.$$

$$y = c_1 e^{-t} + c_2 e^{5t}$$

$$y' = -c_1 e^{-t} + 5c_2 e^{5t}$$

$$\left.\begin{array}{l} y(0) = c_1 + c_2 = 3 \\ y'(0) = -c_1 + 5c_2 = 9 \end{array}\right\} \Rightarrow 6c_2 = 12 \Rightarrow \begin{array}{l} c_2 = 2 \\ c_1 = 1 \end{array}$$

$$y = e^{-t} + 2e^{5t}$$