Determine whether the following statements are true or false...

1. True

   if $Row(A) \neq \mathbb{R}^n \Rightarrow Null(A)$ is not trivial

   $\Rightarrow A$ not invertible

2. False (note: doesn't matter if the underlying field is $\mathbb{C}$ or $\mathbb{R}$)

   $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   invertible but not diagonalizable.

3. False

   $rank(MN) \leq rank(N)$

   for a counter example, Let M be the 0 matrix.

4. True

   $\det(AB) \neq 0 \Rightarrow \det(A) \neq 0$ & $\det(B) \neq 0$

   This proof doesn't work if A & B not square!

5. False

   second order DE $\Rightarrow$ space of solutions is 2 dimensional.

6 Consider the transformation $T: \mathbb{P}_2 \to \mathbb{P}_2$ such that $T(p(x)) = p'(x) + p(x)$.

6a i $T(p_1+p_2) = (p_1+p_2)' + (p_1+p_2)$

$$= p_1' + p_1 + p_2' + p_2 = T(p_1) + T(p_2)$$

ii $T(cp) = (cp)' + cp$

$$= c(p'+p) = cT(p)$$

i & ii $\Rightarrow$ T is LT

6b $T(1) = 0 + 1 = 1 \Rightarrow [T(1)]_\varepsilon = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = 1 + x \Rightarrow [T(x)]_\varepsilon = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$T(x^2) = 2x + x^2 \Rightarrow [T(x^2)]_\varepsilon = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$$[T]_\varepsilon = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

6c This question is just asking if T is invertible.

T^{-1} (aka S) exist $\Leftrightarrow [T]_\varepsilon$ is invertible.

$$\det[T]_\varepsilon = \det\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

Since $\det[T]_\varepsilon = 1 \neq 0 \Rightarrow T^{-1}$ (aka S) exists.

6d $\mathcal{B}$ exist $\Leftrightarrow$ $[T]_\varepsilon$ is diagonalizable.

$$\det([T]_\varepsilon - \lambda I) = (1-\lambda)\cdot\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2, \quad \lambda = 1 \text{ (only e-value)}$$

$\underline{\lambda=1:}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\vec{v} = \vec{0}$$

Only one e-vector $\Rightarrow$ $[T]_\varepsilon$ is not diagonalizable

$\Rightarrow$ desired $\mathcal{B}$ does not exist.

7 Solve the initial value problem,

$$y''(t) + 5y'(t) - 6y(t) = t \quad , \qquad y(0) = 1 \qquad y'(0) = 0 .$$

Homogeneous part: $y'' + 5y' - 6y = 0$

$r^2 + 5r - 6 = 0$

$(r+6)(r-1) = 0 \Rightarrow r = -6, 1$

$y_H = c_1 e^{t} + c_2 e^{-6t}$

Particular solution: $y_p = At + B$ (my guess)

$y_p' = A$

$y_p'' = 0$

$0 + 5A - 6At + B = t$

$-6At + (5A + B) = t \Rightarrow \begin{cases} -6A = 1 \Rightarrow A = -\frac{1}{6} \\ 5A + B = 0 \Rightarrow B = \frac{5}{6} \end{cases}$

General solution: $y = c_1 e^{t} + c_2 e^{-6t} - \frac{1}{6} t + \frac{5}{6}$

$\left.\begin{array}{l} y(0) = c_1 + c_2 + \frac{5}{6} = 1 \\ y'(0) = c_1 - 6c_2 - \frac{1}{6} = 0 \end{array}\right\} \Rightarrow 7c_2 + 1 = 1$

$c_2 = 0 \ , \quad c_1 = \frac{1}{6}$

$y = \frac{1}{6} e^{t} - \frac{1}{6} t + \frac{5}{6}$

Determine whether the following statements are true or false...

8 True

A^TA is symmetric $\Rightarrow$ orthogonally diagonalizable.

9 True

$\text{rank}(A^2) < n \Rightarrow \det(A^2) = 0 \Rightarrow \det(A) = 0$

$\Rightarrow \text{rank}(A) < n$

10 False

The 0 matrix is skew-symmetric

11 True

$(A^{-1})^T$ is the inverse of A^T, but $A^T = A$ so $(A^{-1})^T = A^{-1}$.

12 True

By existence & uniqueness of solutions