Warm Up Exercises.

(1) Let $T: \mathbb{P}_2 \to \mathbb{P}_2$ be defined by $T(p(x)) = xp'(x) + p(1)$. Does there exist a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that $[T]_{\mathcal{B}}$ is diagonal?

Step 1. Compute $[T]_{\mathcal{E}}$.

Step 2. Compute char poly, e-values, e-vector (if nec.)

$$[T]_{\mathcal{E}} = \begin{bmatrix} | & | & | \\ [T(1)]_{\mathcal{E}} & [T(x)]_{\mathcal{E}} & [T(x^2)]_{\mathcal{E}} \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Is this matrix diagonalizable?

$$p(x) = (1-x)(1-x)(2-x)$$
$$= \underbrace{(1-x)^2}_{} \underbrace{(2-x)^1}_{}$$

$\lambda_1 = 1$ is e-value.
At most 2 e-vectors assoc. w/ $\lambda_1 = 1$.

$\lambda_2 = 2$ is a e-value
exactly one e-vector assoc. w/ $\lambda_2 = 2$.

$\downarrow$

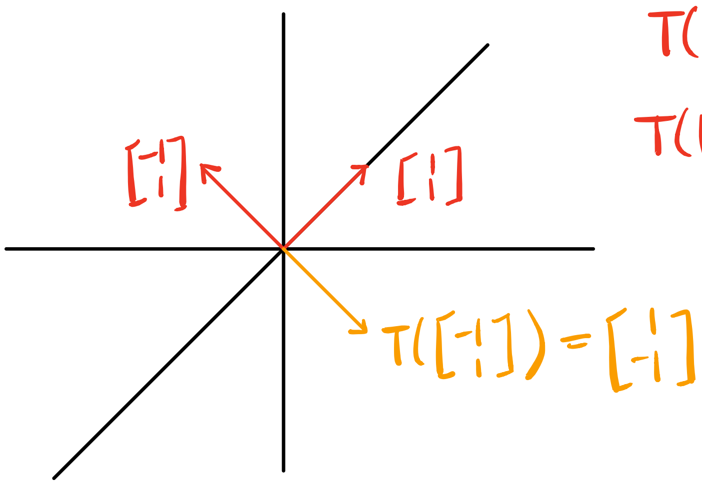
if 2 e-vectors $\Rightarrow [T]_\varepsilon$ diagonalizable
if 1 e-vector $\Rightarrow$ not.

$\lambda_1 = 1$:
$$\begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 1-1 & 0 \\ 0 & 0 & 2-1 \end{bmatrix} \vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Conclusion $[T]_\varepsilon$ only has 2 e-vector
(1 from $\lambda_1 = 1$ & 1 from $\lambda_2 = 2$) $\Rightarrow$ not
diagonalizable $\Rightarrow$ no basis $\mathcal{B}$.

(2) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection about the line $y=x$. Does there exist a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that $[T]_{\mathcal{B}}$ is diagonal? Compute $\mathcal{B}$ if possible.



$T(b_1) = \lambda b_1$

$T(b_2) = \lambda' b_2$

Claim. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are e-vectors of T.

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ e-vector w/ e-value $\lambda = 1$.

$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

e-vector w/ e-value $\lambda = -1$.

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(3) Let $A \in \mathbb{R}^{n \times n}$. If A is similar to an invertible matrix, is A also invertible?

RECALL, A is similar to B if
$$A = PBP^{-1}.$$

$A = PBP^{-1}$ (B is invertible)

A is the product of 3 invertible matrices $\Rightarrow$ A is also invertible.

Inner Product.

not the only
inner product.

Standard inner product on $\mathbb{R}^n$.

$\vec{x}, \vec{y} \in \mathbb{R}^n$ $\quad \vec{x} = (x_1, \dots, x_n)$

$\vec{y} = (y_1, \dots, y_n)$

components of
the vector.

Inner product
or
dot product

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

$$= \vec{x}^T \vec{y}$$

when $\vec{x}$ & $\vec{y}$ are written
as column vectors.

Example. Compute $\vec{x} \cdot \vec{y}$ where

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2(1) + (-1)(3)$$

$$\vec{x} \cdot \vec{y} = -1.$$

Example. Compute $\vec{x} \cdot \vec{y}$ where

$$x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad y = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = 1(-1) + 1(3) + 2(-1)$$

$$\vec{x} \cdot \vec{y} = 0.$$

The length of a vector is,

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$

$\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$.

The distance between two vectors

is

$$d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$$

Two vectors are orthogonal if,

$$\vec{x} \cdot \vec{y} = 0.$$

Example. Compute the distance between $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$

$$d(x, y) = \| x - y \| = \left\| \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} \right\|$$

$$= \sqrt{\begin{bmatrix} 3 & -2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}}$$

$$= \sqrt{9 + 4 + 9}$$

$$d(x, y) = \sqrt{22}$$

Example. Find a vector orthogonal to $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ that is not the zero vector.

Note: $\vec{0}$ is orthogonal to everything.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad x \cdot y = 0$$

$$x \cdot y = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2\underset{\substack{\| \\ -1}}{y_1} + \underset{\substack{\| \\ -1}}{y_2} + 3\underset{\substack{\| \\ 1}}{y_3} = 0 \qquad y = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Example. Is the set S an orthogonal set?

$$S = \left\{ \underset{x}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{y}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}, \underset{z}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \right\}$$

orthogonal set – each vector is orthogonal to all the others.

$x \cdot y = ?$, $x \cdot z = ?$, $y \cdot z = ?$

NOT an orthogonal set.

$$x \cdot y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0$$

$$x \cdot z = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$y \cdot z = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1 \neq 0$$

Example. The set S is orthogonal

normalize the vectors so that S

is an orthonormal set.

↳ set that orthogonal and $\|x\|=1$ for all $x \in S$

$$S = \left\{ \underset{x}{\underset{\|}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}}, \underset{y}{\underset{\|}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}}, \underset{z}{\underset{\|}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}} \right\}$$

$$S' = \left\{ \frac{1}{\|x\|} x, \frac{1}{\|y\|} y, \frac{1}{\|z\|} z \right\}$$

$$\frac{1}{\|x\|} x = \frac{1}{\sqrt{1+1}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\frac{1}{\|y\|} y = \frac{1}{\sqrt{1+1}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\frac{1}{\|z\|} z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example. Show that a linearly dependent set of nonzero vectors cannot be orthogonal.

There exists a maximal linearly independent set $\{v_1, \dots, v_n\}$. Let's assume $\{v_1, \dots, v_n\}$ is an orthogonal set.

$v_{n+1} = a_1 v_1 + \dots + a_n v_n$

In other words $\{v_1, \dots, v_n, v_{n+1}\}$ is linearly dependent. WLOG assume $a_1 \neq 0$.

Claim $v_1 \cdot v_{n+1} \neq 0$.

$$v_1 \cdot v_{n+1} = v_1 \cdot (a_1 v_1 + a_2 v_2 + \dots + a_n v_n)$$

$$= a_1 (v_1 \cdot v_1) + \underbrace{a_2 (v_1 \cdot v_2) + \dots a_n (v_1 \cdot v_n)}_{\text{all 0 by assumption}}$$

$$v_1 \cdot v_{n+1} = a_1 (v_1 \cdot v_1) = 0$$

$$\Rightarrow v_1 = 0 \quad \text{CONTRADICTION}$$

$$v_1 \cdot v_{n+1} = a_1 (v_1 \cdot v_1) \neq 0 \Rightarrow \text{not orthogonal set.}$$