Summary of Computing Eigenvalues and eigenvector.

(1) Compute the characteristic polynomial of the matrix,

$p(x) = \det(A - xI)$ (variable $= x$).

(2) Eigenvalues = roots of the characteristic polynomial.

(3) For each eigenvalue, λ, compute all associated eigenvectors, $\vec{x}$ using the equation,

$(A - \lambda I)\vec{x} = \vec{0}$.

Example. Compute all eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 3 \\ 6 & -6 & -6 \end{bmatrix}$$

$$A - xI = \begin{bmatrix} 2-x & 0 & 0 \\ -1 & 3-x & 3 \\ 6 & -6 & -6-x \end{bmatrix}$$

$$\det(A-xI) = 2-x \cdot \begin{vmatrix} 3-x & 3 \\ -6 & -6-x \end{vmatrix}$$

$$= (2-x)\big((3-x)(-6-x)+18\big)$$

$$= (2-x)(\cancel{-18}+6x-3x+x^2+\cancel{18})$$

$$= (2-x)(x^2+3x)$$

$$= (2-x)\cdot x \cdot (x+3)$$

$\lambda_1 = 0 \Leftrightarrow$ A is not invertible

$\lambda_1 = 0, \ \lambda_2 = -3, \ \lambda_3 = 2.$

$\underline{\lambda_1 = 0}$: $\quad A\vec{x}_1 = \vec{0}$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 3 \\ 6 & -6 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + y \begin{bmatrix} 0 \\ 3 \\ -6 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$\underline{\lambda_2 = -3}$: $\quad (A + 3I)\vec{x}_2 = \vec{0}$

$$\begin{bmatrix} 5 & 0 & 0 \\ 1 & 6 & 3 \\ 6 & -6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x = 0 \Rightarrow x = 0$$

$$\left.\begin{array}{l} \cancel{x} + 6y + 3z = 0 \\ \cancel{6x} - 6y - 3z = 0 \end{array}\right\} \Rightarrow \quad -2y = z$$

$$\vec{x}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$\underline{\lambda_3 = 2}$: $\quad (A - 2I)\vec{x}_3 = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 3 \\ 6 & -6 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x \begin{bmatrix} 0 \\ -1 \\ 6 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} + z \begin{bmatrix} 0 \\ 3 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Diagonalization.

Let $M \in \mathbb{R}^{n \times n}$. The matrix M is diagonalizable if it has n linearly independent eigenvectors.

<u>Claim</u>. Eigenvectors from distinct eigenvalues are linearly independent.

<u>Proof</u>.

M – $n \times n$ matrix

$\lambda_1, \dots, \lambda_j$ distinct e-values

$\vec{x}_1, \dots, \vec{x}_j$ assoc. e-vectors

Suppose $\vec{x}_1, \dots, \vec{x}_j$ are linearly dependent.

There exist a maximal LI subset,

$\vec{x}_1, \dots, \vec{x}_i$ such that

$$a_1\vec{x}_1 + \dots + a_i\vec{x}_i = \vec{x}_{i+1}$$

$$M\left(a_1\vec{x}_1 + \dots + a_i\vec{x}_i\right) = M\vec{x}_{i+1}$$

$$a_1\lambda_1\vec{x}_1 + \dots + a_i\lambda_i\vec{x}_i = \lambda_{i+1}x_{i+1}$$

$$\sum_{m=1}^{i} a_m\lambda_m\vec{x}_m = \lambda_{i+1}\sum_{m=1}^{i} a_m\vec{x}_m$$

$$\sum_{m=1}^{i} \underbrace{a_m(\lambda_m - \lambda_{i+1})} \vec{x}_m = \vec{0}$$

LI set.

all of these must be 0.

But, $\lambda_m \neq \lambda_{i+1}$ for all $m = 1, \dots, i$ $\Rightarrow$ CONTRADICTION.

Consequence, a matrix $M \in \mathbb{R}^{n \times n}$ is diagonalizable if it has n distinct eigenvalues.

Example. Show $M = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ is diagonalizable.

$p(x) = \det(M - xI) = (2-x)x(-1-x)$

$\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 2.$

Since M is 3×3 and it has 3 distinct e-vectors $\Rightarrow$ M is diagonalizable.

Example. Diagonalize the following matrix if possible (compute D and P). If not possible, explain why.

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$p(x) = -(-2+x)^2(-1+x)$$

has one e-vector

$\lambda_1 = 2$ $\quad$ $\lambda_2 = 1$ $\quad$ $\vec{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_1 = 2$ has algebraic multiplicity 2.

$\underline{\lambda_1 = 2}$: $\begin{bmatrix} 2-2 & 0 & 0 \\ 1 & 2-2 & 1 \\ -1 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x + z &= 0 \\ -x - z &= 0 \end{aligned}\right\} \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The matrix is diagonalizable,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

e-vector assoc. $\lambda = 1$

e-vectors assoc. $\lambda = 2$.

Linear Transformations.

From before: given $T: V \to V$ and some basis $\mathcal{B}$ of V we can write a matrix $[T]_{\mathcal{B}}$.

$\dim V = n$
$[T]_{\mathcal{B}}$ is $n \times n$.

Example. Consider the linear transformation $T: \mathbb{P}_2 \to \mathbb{P}_2$ such that $T(ax^2 + bx + c) = 2ax^2 + bx + (c+a)$

Let $\varepsilon = \{1, x, x^2\}$ and $\mathcal{B} = \{1, x, x^2 + 1\}$. Let's compute $[T]_{\varepsilon}$ and $[T]_{\mathcal{B}}$.

$$[T]_{\mathcal{E}} := \begin{bmatrix} | & | & | \\ [T(1)]_{\mathcal{E}} & [T(x)]_{\mathcal{E}} & [T(x^2)]_{\mathcal{E}} \\ | & | & | \end{bmatrix}$$

$$[T(1)]_{\mathcal{E}} = [1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T(x)]_{\mathcal{E}} = [x]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(x^2)]_{\mathcal{E}} = [2x^2+1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [T(x)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(x^2+1)]_{\mathcal{B}} = [2x^2+2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ Diagonal!

Give a transformation $T: V \to V$ we ask the question,

Does there exist a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal?

A basis $\mathcal{B}$ exists if the matrix $[T]_{\varepsilon}$ is diagonalizable. If this is the case, $\mathcal{B}$ is an eigen-basis.

Note: $x \in V$ is an e-vector of $T: V \to V$

if $T(x) = \lambda x$.

Example. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b \end{bmatrix}$. Does there exist a basis β of $\mathbb{R}^2$ such that $[T]_\beta$ is diagonal.

Jordan Block.

$$[T]_\varepsilon = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\lambda = 1$ is the only eigenvalue of T.

$$\begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

T is <u>not</u> diagonalizable

ONLY 1 E-VECTOR $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Example. $T: \mathbb{P}_2 \to \mathbb{P}_2$ is such that $T(p(x)) = xp'(x)$. Does there exist a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that $[T]_{\mathcal{B}}$ is diagonal?

$\mathcal{B} = \{1, x, x^2\}$ is an eigenbasis.

Check ↗ are e-vectors of T,

$T(1) = x \cdot 0 = 0 \quad (\lambda = 0)$

$T(x) = x \cdot 1 = x \quad (\lambda = 1)$

$T(x^2) = x \cdot 2x = 2x^2 \quad (\lambda = 2)$

$$[T]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$