EIGENVALUES & EIGENVECTORS.

I. Definitions.

$A \in \mathbb{R}^{n \times n}$

Def. an eigenvalue/eigenvector pair of a matrix A is a pair $(\lambda, \vec{x})$ where λ is a scalar and $\vec{x} \neq \vec{0}$ is a vector $(\vec{x} \in \mathbb{R}^n)$ such that,

$$A\vec{x} = \lambda\vec{x}$$

II. Characteristic Polynomial.

Def. we define the characteristic polynomial as,

$$p(\lambda) = \det(A - \lambda I)$$

polynomial in λ

Eigenvalues of A are roots of $p(\lambda)$.

$$p(\lambda) = \det(A - \lambda I) = 0.$$

III. How to compute.

(1) Compute e-values by computing roots of $p(\lambda)$.

(2) λ_i denote an eigenvalue. For each eigenvalue I have solve,

$$(A-\lambda_i I)\vec{x} = \vec{0}$$

for $\vec{x}$ (might be multiple).

Example. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(1) $p(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}\right)$

$$= (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$

$$p(\lambda) = (\lambda-3)(\lambda+1) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = -1.$$

(2) $\lambda_1 = 3$. $(A - 3I)\vec{x}_1 = \vec{0}$

any multiple of this is also e-vector.

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x + 2y = 0 \Rightarrow x = y$

$2x - 2y = 0$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ e-vector assoc. w/ $\lambda_1 = 3$.

$\lambda_2 = -1$: $(A + I)\vec{x}_2 = \vec{0}$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x + 2y = 0 \Rightarrow x = -y$

$\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ e-vector assoc. w/ $\lambda_2 = -1$.

IV. Diagonalization.

$A \in \mathbb{R}^{n \times n}$

A matrix, A, is diagonalizable if we can write

$$A = PDP^{-1}$$

D = diagonal matrix, where the diagonal terms are eigenvalues

P = matrix where col's are eigen-vectors (and its invertible)