(1) Compute all eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$\lambda_1 = 2, \quad \lambda_2 = -3$

$\downarrow \qquad\qquad \downarrow$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Automatic, bc A is diagonal.

(2) Compute all eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Compute e-values:

$$p(\lambda) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right)$$

$$= -\lambda(-3-\lambda) + 2$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$$

$$\lambda_1 = -2, \ \lambda_2 = -1.$$

$\lambda_1 = -2$: $\underline{(A+2I)}\vec{x}_1 = \vec{0}$

$\vec{x}_1 \in Nul(A+2I)$ $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{aligned} 2x+y=0 \\ -2x-y=0 \end{aligned}\right\} \Rightarrow y = -2x$$

$\vec{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ e-vector assoc. w/ $\lambda_1 = -2$.

$\lambda_2 = -1$: $(A+I)\vec{x}_2 = \vec{0}$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x + y &= 0 \\ -2x - 2y &= 0 \end{aligned}\right\} \Rightarrow x = -y$$

$\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ e-vector assoc. w/ $\lambda_2 = -1$.

(3) Compute all eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

e-value are diagonal terms

$$p(\lambda) = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right)$$

$$= (2-\lambda)^2 = 0$$

$\lambda = 2$ (only 1 <u>distinct</u> e-value)

$\lambda = 2$: $(A - 2I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y = 0$

$\cancel{0 = 0}$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(4) Compute the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

e-values: $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -1$.

$\lambda_1 = 1$: $(A - I)\vec{x}_1 = \vec{0}$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2y - z = 0$$
$$2y = 0$$
$$-2z = 0$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = 3$: $(A - 3I)\vec{x}_2 = \vec{0}$

$$\begin{bmatrix} -2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z=0$

$$-2x+2y-\cancel{z}=0$$

$$-5z=0$$

$$-2x+2y=0$$

$$x=y$$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3=-1$:

$$\begin{bmatrix} 2 & 2 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x+\cancel{2y}-z=0$$

$$4y=0$$

$$2x-z=0$$

$$z=2x$$

$$\vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

(5) Compute eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 3 & -5 & 5 \\ -2 & 0 & 2 \\ 3 & -3 & 5 \end{bmatrix}$$

$$p(\lambda) = -(-8+\lambda)(-2+\lambda)(2+\lambda)$$

$$\lambda_1 = -2, \quad \lambda_2 = 2, \quad \lambda_3 = 8$$

$\lambda_1 = -2$:
$$\begin{bmatrix} 5 & -5 & 5 \\ -2 & 2 & 2 \\ 3 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x - 5y + 5z = 0$$
$$-2x + 2y + 2z = 0$$
$$3x - 3y + 7z = 0$$

$$\cancel{x} - \cancel{y} + z = 0$$
$$-\cancel{x} + \cancel{y} + z = 0$$
$$2z = 0 \Rightarrow z = 0$$

After setting $z=0$ all 3 eq. give the same relation $x=y$.

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(6) (True or False) For a matrix $A \in \mathbb{R}^{n \times n}$ the # of distinct eigenvalues is equal to the number of eigenvectors.

False. In general

distinct e-values $\leq$ # e-vectors

Can have cases where

e-vectors $>$ # e-values.

For example,

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

A has 1 distinct e-value & 2 e-vectors.

(17) (True or False) If $\lambda = 0$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ then A is not invertible.

True. $\det(A - \lambda I) = 0$

($\lambda = 0$ evalue) $\det(A) = 0$

$$\Updownarrow$$

A is not invertible.

(8) (True or False) Any invertible matrix is diagonalizable.

False.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ invertible & diag.

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ not diag & invertible

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ diag & not invertible

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ not diag & not invertible

(9) (True or False) If $A, B \in \mathbb{R}^{n \times n}$ have the same set of eigen-values, then $A = B$.

False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

Have same e-value, diff. matrices.

(10) (True or False) If $A, B \in \mathbb{R}^{n \times n}$ are diagonalizable with the same set of eigenvalues, then there exists an invertible matrix P such that $A = PBP^{-1}$.

True.

A & B are similar matrices.

$$A = P_1 D P_1^{-1} \qquad B = P_2 D P_2^{-1}$$

$$P_2^{-1} B = D P_2^{-1}$$

$$P_2^{-1} B P_2 = D$$

$$A = P_1 P_2^{-1} B P_2 P_1^{-1}$$

$$P = P_1 P_2^{-1}$$