Full Singular Value Decomposition.

<u>Def.</u> For any m×n matrix A we can compute,

$$\underset{m\times n}{A} = \underset{m\times m}{U} \cdot \underset{m\times n}{\Sigma} \cdot \underset{n\times n}{V^T}$$

Where U and V are orthogonal matrices and Σ is a "diagonal" matrix of singular values.

Example 1. $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$

$$A^TA = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

have almost the same eigenvalues.

$$AA^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

Eigenvalues of A^TA:

$\lambda = a \pm b = 17 \pm 8 \qquad \lambda_1 = 25 \qquad \lambda_2 = 9$

$\sigma_1 = \sqrt{25} = 5 \qquad \sigma_2 = \sqrt{9} = 3$

$$\Sigma_1 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

$V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$ where v_1 & v_2 are eigenvectors of A^TA.

Compute eigenvectors of A^TA.

$\lambda_1 = 25$: $\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 9$: $\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{array}{cccc} & \lambda_1 = 25 & \lambda_2 = 9 & \lambda_3 = 0 \\ U = & \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix} & & \end{array}$$

where $u_1, u_2,$ & u_3 are eigenvectors of AA^T.

$$\lambda_1 = 25: \quad \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 9: \quad \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

row 1 + row 2: $16x + 16y = 0 \quad x = 1, \ y = -1$

row 3: $2(1) - 2(-1) = z \Rightarrow z = 4$.

$$u_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = 0: \quad \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} u_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$25x + 25y = 0 \Rightarrow \quad x = \overset{2}{1} \quad y = -\overset{-2}{1}$$

$$2 + 2 + 8z = 0 \Rightarrow \quad z = \frac{1}{2} \;\; {}^{1}$$

$$u_3 = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \qquad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & 1/3 \end{bmatrix}$$

Steps for computing full SVD:

1. Compute eigenvectors & eigenvalues of A^TA. For eigenvalue λ_i, corresponding singular value is,

   $\sigma_i = \sqrt{\lambda_i}$.

   $V =$ ordered orthonormal eigenbasis of A^TA.

2. Compute eigenvectors of AA^T.

   $U =$ ordered orthonormal eigenbasis of AA^T.

(*) Might be quicker to compute eigenvalues using AA^T.

Example 2. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$$

Eigenvalues $\Rightarrow \lambda = 6 \pm (-2)$ $\quad \Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

$$\lambda_1 = 8 \quad \lambda_2 = 4$$

Compute eigenvectors of AA^T = columns of U!

$$\lambda_1 = 8: \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 4: \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} u_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigenvector of A^TA = columns of V.

$$A^TA = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 8 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = 8$: $\begin{bmatrix} -6 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -6 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_2 = 4$: $\begin{bmatrix} -2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & -2 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 0: \begin{bmatrix} 2 & 0 & 2 \\ 0 & 8 & 0 \\ 2 & 0 & 2 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\Sigma_1 = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

FUN FACTS!

1. Will the eigenvalues of A^TA for $A \in \mathbb{R}^{m \times n}$ be non-negative? YES!

Claim: A^TA is positive semi-definite.

WTS: $u^T(A^TA)u \geq 0 \quad \forall u \in \mathbb{R}^n$

$$(Au)^T(Au) = \|Au\|^2 \geq 0$$

usual inner product norm.

$\Rightarrow A^TA$ is pos. semi-def $\Rightarrow$ eigenvalues of A^TA are non-negative.

$$u^T A A^T u = \|A^T u\|^2 \geq 0$$

2. The non-zero eigenvalues of A^TA are the same as the non-zero eigenvalues of AA^T.

μ = non-zero eigenvalue of A^TA.

$$A^TAx = \mu x$$

what if $Ax = 0$?
$\Rightarrow A^TAx = 0 \Rightarrow \mu = 0$.

$$AA^T(Ax) = \mu(Ax)$$

$\Rightarrow \mu$ is also an eigenvalue of AA^T.

Example 3. $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$

$$\det \begin{bmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{bmatrix} = -1+\lambda^2-3 = \lambda^2-4$$

$\lambda_1 = 2$, $\lambda_2 = -2$

$\lambda = 2$: $\begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$ $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

not orthogonal

CAN'T USE G-S.

$\lambda = -2$: $\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & 2 \\ 2 & 10-\lambda \end{bmatrix} = 20-12\lambda+\lambda^2-4$$

$$\det(A-\lambda I) = \lambda^2 - 12\lambda + 16$$

$$\lambda = \frac{12 \pm \sqrt{144-64}}{2} = 6 \pm 2\sqrt{5}$$

$$\sigma_1 = \sqrt{6+2\sqrt{5}} \quad , \quad \sigma_2 = \sqrt{6-2\sqrt{5}}$$

$U \neq V$.