ORTHOGONAL MATRICES.

Def. An orthogonal matrix is a square matrix with orthonormal columns.

i. $QQ^T = Q^TQ = I$.

ii. $\cancel{Q^{-1} = Q^T}$.

Columns:
- orthogonal
- normalized.

i.e. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ orthogonal matrix

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ not orthogonal matrix

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ orthogonal matrix.

Try the following questions on your own,

1. Show that the matrix M is orthogonal.

$$M = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad \theta \in \mathbb{R}$$

2. Compute M^{-1} (for M above).

3. Prove that $A = I - 2aa^T$ is orthogonal, where a is a unit vector.

4. (T or F) The product of orthogonal matrices is orthogonal.

5. (T or F) Let A and B be $n \times n$ orthogonal matrices. Is $C = \frac{1}{2}(A+B)$ orthogonal?

Solutions:

1. $\left\{ u_1 = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{bmatrix} \right\}$

Shows vectors are orthogonal

$u_1 \cdot u_2 = 0, \quad u_2 \cdot u_3 = 0$

$$u_1 \cdot u_3 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$$

$$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

WTS vectors are unit length

$\|u_2\|^2 = 1.$

$$\|u_1\|^2 = \cos^2\theta + \sin^2\theta \overset{\text{trig}}{=} 1.$$

$$\|u_3\|^2 = \sin^2\theta + \cos^2\theta = 1.$$

$\Rightarrow$ M is orthogonal.

2. $M^{-1} = M^T = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

Show $AA^T = I \Rightarrow A$ is orthogonal.

3. $AA^T = (I - 2aa^T)(I - 2aa^T)^T$

$= (I - 2aa^T)(I - 2aa^T)$ (*) $A^T = A$

$= I - 2aa^T - 2aa^T + 4a\underbrace{a^Ta}_{a^Ta=1}a^T$

$= I - 4\cancel{a}a^T + 4\cancel{a}a^T = I.$

4. TRUE

same dimensions.

$A_1, \dots, A_n$ be orthogonal

$AA^T = I \Rightarrow A$ is orthogonal

$(A_1 \cdots A_n)(A_1 \cdots A_n)^T = A_1 \cdots \underbrace{\cancel{A_n A_n^T}}_{I} \cdots A_1^T$

$= A_1 A_1^T = I.$

$A_1 \cdots A_n$ is orthogonal.

5. FALSE.

$$
\begin{aligned}
CC^T &= \tfrac{1}{2}(A+B)\tfrac{1}{2}(A+B)^T \\
&= \tfrac{1}{4}(A+B)(A^T+B^T) \\
&= \tfrac{1}{4}\big(\underbrace{AA^T}_{= I} + \underbrace{BA^T}_{= ?} + \underbrace{AB^T}_{= ?} + \underbrace{BB^T}_{= I}\big)
\end{aligned}
$$

Counter example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\tfrac{1}{2}(A+B) = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

col. are not orthogonal.

(REAL) SYMMETRIC MATRICES.

<u>Def.</u> A matrix $A \in \mathbb{R}^{n \times n}$ is

symmetric if $A = A^T$.

Example 1. Let $A \in \mathbb{R}^{n \times n}$. Show that

$A + A^T$ is symmetric.

$$(A + A^T)^T = A^T + (A^T)^T$$
$$= A^T + A = A + A^T.$$

$(A + A^T)^T = A + A^T \Rightarrow A + A^T$ is sym.

Example 2. Let $A \in \mathbb{R}^{n \times n}$. Is $A - A^T$ a symmetric matrix?

$$(A - A^T)^T = A^T - A$$

let $B = A - A^T$

$$= -(A - A^T)$$

$\Rightarrow B^T = -B$ NOT SYM.

↑

Skew-symmetric matrix.

PROPERTIES OF REAL SYM. MATRICES.

A is real

A is symmetric.

$A \in \mathbb{R}^{n \times n}$, $A = A^T$.

① A is **always** diagonalizable with real eigenvalues.

② Eigenvectors can **always** be chosen to form an orthogonal matrix.

there exist an ortho-normal eigenbasis.

So we can write, $A = VDV^T$.

Example 3. Find an orthonormal eigenbasis of M.

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

1. Find eigenvalues.

$$\det(M-\lambda I) = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2-3\lambda+2-1)+(\lambda-1)$$

$$= (1-\lambda)(\lambda^2-3\lambda+1-1)$$

$$= (1-\lambda)(\lambda^2-3\lambda)$$

$$= (1-\lambda)\lambda(\lambda-3)$$

3 distinct eigenvalues - $\{1, 0, 3\}$

2. Find eigenvectors:

$\lambda_1 = 0$: $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\lambda_2 = 1$: $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$\lambda_3 = 3$: $$\begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -2x + z &= 0 \\ -2y + z &= 0 \end{aligned}\right\} \Rightarrow x = y$$

$$x + y - z = 0$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$v_1 \cdot v_2 = 0$, $v_2 \cdot v_3 = 0$, $v_1 \cdot v_3 = 0$.

Just need to normalize.

$$v_1' = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \qquad v_2' = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \qquad v_3' = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$V = \begin{bmatrix} | & | & | \\ v_1' & v_2' & v_3' \\ | & | & | \end{bmatrix}$$ is orthogonal.

Example 4. Find an orthonormal eigen basis of S.

$$S = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix}$$

1. Compute eigenvalues.

$$\begin{vmatrix} 1-\lambda & 0 & \sqrt{2} \\ 0 & 2-\lambda & 0 \\ \sqrt{2} & 0 & -\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & \sqrt{2} \\ \sqrt{2} & -\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2-\lambda-2)$$

$$= (2-\lambda)(\lambda-2)(\lambda+1)$$

Eigenvalues $- \{-1, 2\}$

repeated.

Check eigenvectors of $\lambda = 2$.

$$\begin{bmatrix} -1 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

orthogonal.

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

3. Check $\lambda = -1$.

$$\begin{bmatrix} 2 & 0 & \sqrt{2} \\ 0 & 3 & 0 \\ \sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

NORMALIZE!

ALTERNATIVE INNER PRODUCT.

$A \in \mathbb{R}^{n\times n}$

A symmetric & positive definite.

$\lambda_i > 0$.

Then $\langle u, v\rangle = u^T A v$ is an inner product on $\mathbb{R}^n$.

For usual $\langle \cdot, \cdot \rangle$
$A = I$.

Example 5. Is $\langle u, v\rangle = u^T A v$ an inner product on $\mathbb{R}^2$?

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

① Is A symmetric? YES.

③ Is A pos.-def.? NO.

Compute eigenvalues...

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 - 9$$

$$= \lambda^2 - 2\lambda - 8$$

$$= (\lambda - 4)(\lambda + 2)$$

$$\lambda = 4, -2$$

A is not pos. def.
bc $-2 \leq 0$.

$\langle u, v \rangle = u^T A v$ is NOT an inner product.

Example 6. What are the conditions on a & b such that A is positive definite.

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = \lambda^2 - 2a\lambda + a^2 - b^2$$

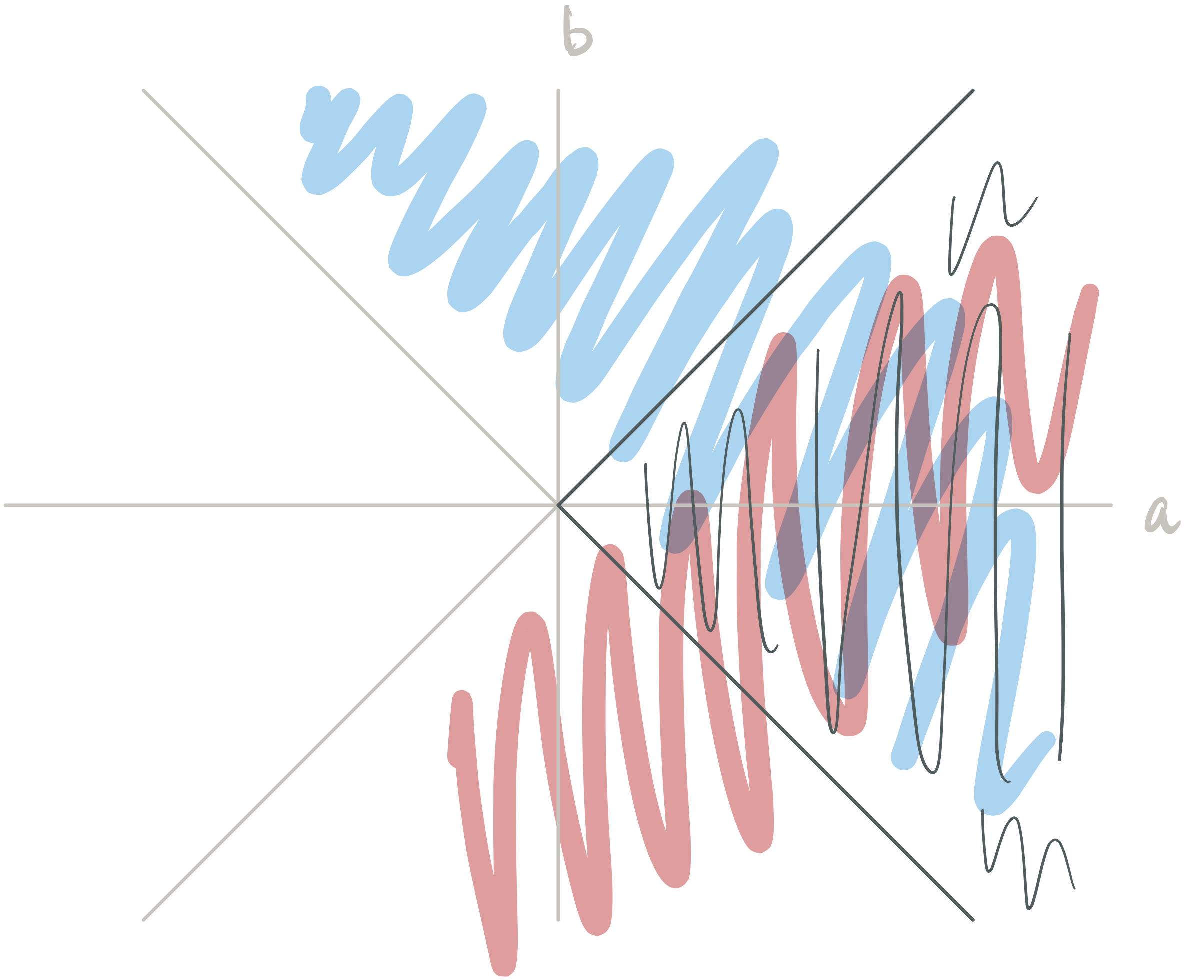
$$\lambda = \frac{2a \pm \sqrt{\cancel{4a^2} - 4(\cancel{a^2} - b^2)}}{2}$$

$$= \frac{2a \pm \sqrt{4b^2}}{2} = \frac{2a \pm 2b}{2}$$

$$\lambda = a \pm b.$$

$a+b>0$ & $a-b>0$

$\boxed{a>-b}$ $\boxed{a>b}$



$a > 0$

$|a| > |b|$ } another interpretation.