QUIZ REVIEW.

1. Compute the singular values of the following matrix:

$$A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4 & 2 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & -10 \\ -10 & 5 \end{bmatrix}$$

compute the eigenvalues.

$$\det \begin{bmatrix} 20-\lambda & -10 \\ -10 & 5-\lambda \end{bmatrix} = \lambda^2 - 25\lambda + 100 - 100$$

$$= \lambda^2 - 25\lambda = \lambda(\lambda - 25)$$

$$\lambda = 25, 0 \Rightarrow \sigma_1 = 5, \sigma_2 = 0.$$

2. Compute the eigenvalues and singular values for the following matrices:

i. $\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ ii. $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ iii. $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

i. $\det \begin{bmatrix} 2-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda)$

$\lambda = 2, 1.$

$$\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\det \begin{bmatrix} 4-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix} = \lambda^2 - 9\lambda + 20 - 16$$

$$= \lambda^2 - 9\lambda + 4$$

$$\lambda = \frac{9 \pm \sqrt{65}}{2}, \quad \boxed{\sigma = \sqrt{\frac{9 \pm \sqrt{65}}{2}}}$$

ii. $\det \begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 - 9$

$$= \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2)$$

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, $\lambda = a \pm b.$

$\lambda = 4, -2$

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

$$\det\begin{bmatrix} 10-\lambda & 6 \\ 6 & 10-\lambda \end{bmatrix} = \lambda^2 - 20\lambda + 100 - 36$$

$$= \lambda^2 - 20\lambda + 64$$

$$\lambda = 4, 16$$

$$\sigma = 2, 4$$

iii. $$\det\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3$$

$$= (\lambda-3)(\lambda-1)$$

$$\lambda = 3, 1$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\det\begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix} = \lambda^2 - 10\lambda + 9$$

$$= (\lambda-9)(\lambda-1)$$

$$\lambda = 9, 1$$

$$\sigma = 3, 1.$$

3. Determine whether the following statements are true or false:

i. Let A be an $n \times n$ matrix. If A is orthogonally diagonalizable, then A is symmetric.

ii. If A is a symmetric $n \times n$ matrix, then $\langle u, v \rangle = u^T A v$ is an inner product.

iii. For any real matrix A, the SVD of A exists and is unique.

i. True

$V^{-1} = V^T$

Let $A = VDV^{-1}$
$= VDV^T$

because orthogonally diagonalizable.

Then $A^T = (VDV^T)^T = (V^T)^T D^T V^T$
$= VDV^T = A$

$\square$

to prove sym show $A^T = A$.

ii. False.

$\lambda > 0 \;\forall \lambda$ eigenvalue.

A must be positive definite. A counter example would be...

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

<u>Exercise.</u> show $\langle u, v \rangle = u^T A v$ is not an inner product.

iii. False.

SVD is <u>not</u> unique (just switch the ordering of the singular values).

4. Solve for the general solution to the following differential equation.

$$y''(t) - y'(t) - 2y(t) = e^t$$

Homogeneous eq: $y'' - y' - 2y = 0$

aux eq: $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0$$

$$r = 2, -1.$$

$$y_H(t) = c_1 e^{2t} + c_2 e^{-t}$$

Solve for A.

Form of particular solution: $y_p(t) = Ae^t$

$$y_p'' - y_p' - 2y_p = Ae^t - Ae^t - 2Ae^t = e^t$$

$$-2Ae^t = e^t \Rightarrow A = -\frac{1}{2}$$

$$y(t) = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2}e^t$$

5. Solve the following initial value problem,

$$y''(t) - y'(t) - 6y(t) = 0 \;, \quad y(0) = 1 \;, \quad y'(0) = 3$$

aux eq: $r^2 - r - 6 = 0$

$(r-3)(r+2) = 0$

$r = 3, -2$

$y(t) = c_1 e^{3t} + c_2 e^{-2t}$

$y(0) = 1 = c_1 + c_2$

Solve for c_1 & c_2

$y'(0) = 3 = 3c_1 - 2c_2$

$y(t) = e^{3t}$ $\quad (c_1 = 1, \; c_2 = 0)$

6. Determine whether the following statements are true or false.

i. Let $y_1(t)$ be a solution of $y''(t)+3y'(t)=0$, then $3y_1(t)$ is also a solution.

ii The particular solution to the differential equation $y''-3y'-4y=te^t$ has the form $(A_1t+A_0)e^t$ $(A_0, A_1 \in \mathbb{R})$.

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iii. The differential equation $y'''+y''-2y'+y=0$ has 3 linearly independent solutions.

e^t

i. True

Let's prove it...

Is this true if non-homogeneous?

$$3y_1''+3\cdot 3y_1' = 3(y_1''+3y_1')$$
$$= 3\cdot 0 = 0. \quad \square$$

ii. True

Check the homogeneous sol'n:

$r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0 \quad r = 4, -1. \neq 1$

iii. True

3rd order derivative $\Rightarrow$ 3 LI sol'n.

This is still true if non-homogeneous.