WARM UP PROBLEMS.

Convert the following higher order ODEs to systems of first order ODEs:

1. $y'''(t) - 2y''(t) - 3y'(t) = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix} \qquad \frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$$

2. $y'''(t) - a_2 y''(t) - a_1 y'(t) - a_0 y(t) = 0.$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix}$$

Example 1. $y'''(t) - 2y''(t) - 3y'(t) = 0$

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}}_{\parallel \atop A}\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$$

① Solution 1.

$y''' - 2y'' - 3y' = 0 \Rightarrow$ aux. eq.

$$r^3 - 2r^2 - 3r = 0$$
$$r(r^2 - 2r - 3) = 0$$
$$r(r-3)(r+1) = 0$$
$$r = 0, 3, -1$$

$$y(t) = c_1 + c_2 e^{3t} + c_3 e^{-t}$$

② Solution 2.

$$\det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 3 & 2-\lambda \end{bmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 2\lambda - 3)$$

$$= -\lambda(\lambda - 3)(\lambda + 1) = 0$$

$\lambda = 0: \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \lambda = 3: \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$

$\lambda = -1: \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

three LI sol'n

$$\vec{y}(t) = c_1\vec{v}_1 + c_2\vec{v}_2 e^{3t} + c_3\vec{v}_3 e^{-t}.$$

IVP: $y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1.$

$$y(t) = c_1 + c_2 e^{3t} + c_3 e^{-t}$$

$$y(0) = c_1 + c_2 + c_3 = 1.$$

$$y'(t) = 3c_2 e^{3t} - c_3 e^{-t}$$

NOT GREAT :(

will take a while.

$$y'(0) = 3c_2 - c_3 = 0$$

$$y''(t) = 9c_2 e^{3t} + c_3 e^{-t}$$

$$y''(0) = 9c_2 + c_3 = 1$$

FUNDAMENTAL MATRIX.

Suppose we have a system of ODEs,

vector with n entries.

$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t)$. There are n linearly independent

n×n matrix (constant for now).

solutions; $\vec{y}_1(t), \ldots, \vec{y}_n(t)$.

are vectors!

Fundamental matrix: $Y(t) = \begin{bmatrix} | & | & & | \\ \vec{y}_1(t) & \vec{y}_2(t) & \cdots & \vec{y}_n(t) \\ | & | & & | \end{bmatrix}$

easy to find $\tilde{Y}(t)$ s.t $\frac{d}{dt}\tilde{Y}(t) = A\tilde{Y}(t)$.

(*) $\underline{Y(0) = I_n}$ ← work

Facts about the fundamental matrix:

$\vec{y}_G(t) = Y(t)\vec{c}$

$\vec{c}$ = initial conditions

vector of coefficients $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\frac{d}{dt}Y(t) = AY(t)$.

Example 2. Compute the fundamental matrix of the ODE, $y''(t) - y(t) = 0$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$$

$\lambda = 1, -1.$

$\lambda = 1: \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ $\lambda = -1: \vec{v}_{-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The DE, $\frac{d}{dt}\vec{y} = A\vec{y}$, has 2 LI solutions,

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t , \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \right\}$$

e^t & e^{-t} are LI b/c
$c_1 e^t + c_2 e^{-t} = 0$ $(\forall t \in \mathbb{R})$
iff $c_1 = c_2 = 0$.

$$\tilde{Y}(t) = \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}$$

<u>Check</u>. $\frac{d}{dt}\tilde{Y}(t) = A\tilde{Y}(t)$.

<u>Problem</u>. $\tilde{Y}(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \neq I_2$

It is easy to compute $\tilde{Y}(t)$ such that

$\frac{d}{dt}\tilde{Y}(t) = A\tilde{Y}(t)$, but it not always the case that

$\tilde{Y}(0) = I_n$.

$$\boxed{Y(t) = \tilde{Y}(t)\tilde{Y}(0)^{-1}.}$$

$Y(0) = \tilde{Y}(0)\tilde{Y}(0)^{-1} = I_n$

$$\tilde{Y}(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} \qquad \tilde{Y}(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\tilde{Y}(0)^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Fundamental matrix.

$$Y(t) = \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix}$$

Example 3. Compute the fundamental matrix of the ODE in example 1.

Three LI sol'n: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}e^{-t}, \begin{bmatrix}1\\3\\9\end{bmatrix}e^{3t}$.

$$\tilde{Y}(t) = \begin{bmatrix} 1 & e^{-t} & e^{3t} \\ 0 & -e^{-t} & 3e^{3t} \\ 0 & e^{-t} & 9e^{3t} \end{bmatrix}$$

$$\tilde{Y}(0) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & 9 \end{bmatrix} \qquad \tilde{Y}(0)^{-1} = \begin{bmatrix} 1 & 2/3 & -1/3 \\ 0 & -3/4 & 1/4 \\ 0 & 1/12 & 1/12 \end{bmatrix}$$

$Y(t) = \tilde{Y}(t)\tilde{Y}(0)^{-1}$. (*) recompute IVP.

Example 4. Compute the fundamental matrix of

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\vec{y}(t).$$

DOES NOT come from higher order DE.

$$\det\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = \lambda^2 - 5\lambda + 4 + 2$$

$$= \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2)$$

$$\lambda = 3, 2.$$

$$\lambda = 3: \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2: \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2 LI Sol'n: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

$$\tilde{Y}(t) = \begin{bmatrix} e^{3t} & 2e^{2t} \\ e^{3t} & e^{2t} \end{bmatrix} \quad \tilde{Y}(0) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\tilde{Y}(0)^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$Y(t) = \begin{bmatrix} e^{3t} & 2e^{2t} \\ e^{3t} & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2t} - e^{3t} & 2e^{3t} - 2e^{2t} \\ e^{2t} - e^{3t} & 2e^{3t} - e^{2t} \end{bmatrix}$$

IVP. $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ $y_1(0) = 3$, $y_2(0) = -1$.

$$\vec{y}(t) = Y(t) \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2e^{2t} - e^{3t} & 2e^{3t} - 2e^{2t} \\ e^{2t} - e^{3t} & 2e^{3t} - e^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8e^{2t} - 5e^{3t} \\ 4e^{2t} - 5e^{3t} \end{bmatrix}$$

Example 5. $y''(t) - 2y'(t) + y(t) = 0$, $y(0) = 1$, $y'(0) = -1$.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

$\lambda = 1$ (only 1!)

ONLY 1 E-VECTOR!

Use Aux. Eq!

$$y'' - 2y' + y = 0 \Rightarrow r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \Rightarrow r = 1.$$

$$y(t) = c_1 e^t + c_2 t e^t.$$

$$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t.$$

2 LI sol'n: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t, \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^t$

$$\tilde{Y}(t) = \begin{bmatrix} e^t & te^t \\ e^t & e^t + te^t \end{bmatrix} \qquad \tilde{Y}(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\tilde{Y}(0)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$Y(t) = \begin{bmatrix} e^t - te^t & te^t \\ -te^t & e^t + te^t \end{bmatrix}$$

$$\vec{y}(t) = \begin{bmatrix} e^t - te^t & te^t \\ -te^t & e^t + te^t \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{y}(t) = \begin{bmatrix} e^t - 2te^t \\ -e^t - 2te^t \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$$

Check $\begin{bmatrix} y \\ y' \end{bmatrix}$ ...

$$(e^t - 2te^t)' = e^t - 2e^t - 2te^t$$
$$= -e^t - 2te^t \checkmark$$

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\vec{y}(t).$$

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

$$\left.\begin{aligned} y_1'(t) &= y_1(t) + 2y_2(t) \\ y_2'(t) &= -y_1(t) + 4y_2(t). \end{aligned}\right\}$$