WARM UP PROBLEMS.

1. Find the general solution to $y''(t) - y'(t) = 0$.

aux. eq. $r^2 - r = 0$

$r(r-1) = 0 \Rightarrow r = 1, 0$

$$y(t) = c_1 e^{0t} + c_2 e^t = c_1 + c_2 e^t.$$

2. Write down the form of the particular solution to $y''(t) - y'(t) = te^t$.

poly of deg 1 $(A_0 + A_1 t)$

exp. (e^t)

$y_p(t) = (A_0 + A_1 t)e^t$ X $A_0 e^t$ is part of the general solution.

$$\boxed{y_p(t) = (A_0 t + A_1 t^2) e^t}$$

3. Solve for the general solution to
$y'''(t) + 3y''(t) - 4y'(t) - 12y(t) = 0.$

aux. eq. $r^3 + 3r^2 - 4r - 12 = 0$

$$(r^2 - 4)(r+3) = 0$$

$$(r+2)(r-2)(r+3) = 0$$

$$r = -2, 2, -3.$$

$$y(t) = c_1 e^{-2t} + c_2 e^{2t} + c_3 e^{-3t}.$$

WRITING DEs AS MATRIX EQUATIONS.

Example 1. $y''(t) - y'(t) = 0. \Rightarrow y'' = y'$

2nd order DE $\Rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{= A} \begin{bmatrix} y \\ y' \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda(1-\lambda) - 0 = \lambda^2 - \lambda$$
$$= \lambda(\lambda - 1) = 0$$

the same as the auxilary equation

Example 2. $y'''(t)+3y''(t)-4y'(t)-12y(t)=0.$

3rd order $\Rightarrow \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 4 & -3 \end{bmatrix}\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$$

$$\det\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 12 & 4 & -3-\lambda \end{bmatrix} = -\lambda\begin{vmatrix} \lambda & 1 \\ 4 & -3-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 12 & -3-\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2+3\lambda-4)+12$$

$$= -\lambda^3-3\lambda^2+4\lambda+12=0$$

$$\lambda = 2, -2, -3.$$

$\lambda=2$: $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 12 & 4 & -5 \end{bmatrix}$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$\lambda = -2:$ $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 12 & 4 & -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

$\lambda = -3:$ $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 12 & 4 & 0 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = c_1\vec{v}_1 e^{2t} + c_2\vec{v}_2 e^{-2t} + c_3\vec{v}_3 e^{-3t}$$

Example 3. $x'''(t) - x'(t) = 0$

$$\frac{d}{dt}\begin{bmatrix} x \\ x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \\ x''' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ x' \\ x'' \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1)$$

$$= -\lambda(\lambda+1)(\lambda-1)$$

$$\lambda = 0, \lambda = -1, \lambda = 1$$

$$x(t) = c_1 + c_2 e^{t} + c_3 e^{-t}$$

$$\vec{v}_{\lambda=0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_{\lambda=1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_{\lambda=-1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Example 4. $r'(t) = ar(t) + bf(t)$
$f'(t) = cr(t) + df(t)$

case 1: $a=0,\ b=1,\ c=1,\ d=0$

case 2: $a=1,\ b=4,\ c=-1,\ d=1$

Case 1. $r'(t) = f(t)$

$f'(t) = r(t)$

$$\frac{d}{dt}\begin{bmatrix} r(t) \\ f(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} r(t) \\ f(t) \end{bmatrix}$$

$$\det\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = +\lambda^2 - 1 = (\lambda-1)(\lambda+1) = 0$$

$\lambda = 1,\ \lambda = -1$

$$\vec{v}_{\lambda=1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v}_{\lambda=-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} r \\ f \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

<u>Case 2</u>:

$$r'(t) = r(t) + 4f(t)$$
$$f'(t) = -r(t) + f(t)$$

$$\frac{d}{dt}\begin{bmatrix} r \\ f \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} r \\ f \end{bmatrix}$$

$$\begin{bmatrix} r(t) \\ f(t) \end{bmatrix} = \begin{bmatrix} r \\ f \end{bmatrix} e^{\cdots}$$

$$\det \begin{bmatrix} 1-\lambda & 4 \\ -1 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 - 4$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3 \quad \lambda = -1$$

$\lambda = 3$: $\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = -1$: $\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} r \\ f \end{bmatrix} = c_1 \vec{v}_1 e^{3t} + c_2 \vec{v}_2 e^{-t}.$$

$r(t)$ ——.

$f(t)$ ——.

WORKSHEET PROBLEMS:

Write down the matrix eq. for the following DEs:

1. $y''(t) + 3y'(t) - y(t) = 0$

$$(1)\ \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

2. $x''(t) + x(t) = 0$

$$(2)\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

3. $x'''(t) + 2x''(t) - x(t) = 0$

$$(3)\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

4. $\sum_{i=0}^{n} y^{(i)}(t) = 0$

5. $x'(t) = Ax(t) + By(t)$
   $y'(t) = Cx(t) + Dy(t)$

$$(5)\ \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$(4)\ \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix}$$

consider $\sum_{i=0}^{n} a_i y^{(i)}(t) = 0$, $a_n \neq 0$.