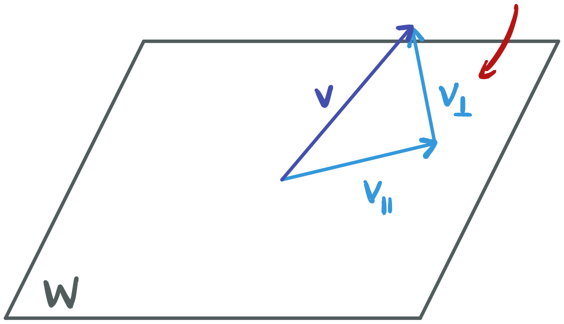
Orthogonal Projections.

"shortest distance" from v to W.



$v_\parallel = \mathrm{Proj}_W v \in W$

$v_\perp = v - \mathrm{Proj}_W v \in W^\perp$

$\forall w \in W, \; v_\perp \cdot w = 0$

Example 1. Compute the orthogonal projection of $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ onto the subspace $W = \mathrm{Span}\left\{ \overset{\text{"}u\text{"}}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} \right\}$.

$$\mathrm{Proj}_W v = \mathrm{Proj}_u v = \frac{v \cdot u}{u \cdot u} u.$$

$u \cdot u = \|u\|^2$

$$= \frac{\begin{bmatrix}1\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\1\end{bmatrix}}{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\1\end{bmatrix}}\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix}2\\1\\1\end{bmatrix} = v_{\parallel}$$

(*) Show $\mathrm{Proj}_W v$ is the same when diff basis elt chosen.

$$u = \begin{bmatrix}4\\2\\2\end{bmatrix}$$

Example 2. Compute the orthogonal projection of $v = \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$ onto the subspace $W = \mathrm{Span}\left\{ \underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}_{=u_1}, \underbrace{\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}}_{=u_2} \right\}$.

↑ this is a basis.
this is an orthogonal basis.

$$v_{\parallel} = \frac{v\cdot u_1}{u_1\cdot u_1}u_1 + \frac{v\cdot u_2}{u_2\cdot u_2}u_2$$

(*) this formula won't work if my basis is not orthogonal.

$$v_{\parallel} = \frac{\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}{\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \frac{\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}}{\begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \frac{-1}{2}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix} = v_{\parallel}$$

$$v_{\perp} = v - v_{\parallel} = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} - \begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix} = \begin{bmatrix}0\\\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix}$$

$v_{\perp}$ is orthogonal to $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$

Example 3. Compute the projection of $v = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ onto the subspace Col(A)

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$.

basis of Col(A) $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$

these are orthogonal.

$$v_{\parallel} = \frac{\begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \frac{\begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$v_{\parallel} = \frac{6}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 0 = \frac{6}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

What if we don't have an orthogonal basis for the subspace?

Not orthogonal basis of W $\xrightarrow[\text{process.}]{\text{Gram-Schmidt}}$ orthogonal basis of W

How does it work?

Let $\{u_1, u_2, \dots, u_n\}$ be a basis of W. We would like to create a new basis $\{u_1', u_2', \dots, u_n'\}$ that is orthogonal.

first element of my new basis.

1. $u_1' = u_1$

2. $$u_2' = u_2 - \operatorname{Proj}_{u_1'} u_2$$

$$= u_2 - \frac{u_2 \cdot u_1'}{u_1' \cdot u_1'} \cdot u_1'$$

new old.

3. $$u_3' = u_3 - \operatorname{Proj}_{u_2'} u_3 - \operatorname{Proj}_{u_1'} u_3$$

$$= u_3 - \frac{u_3 \cdot u_2'}{u_2' \cdot u_2'} u_2' - \frac{u_3 \cdot u_1'}{u_1' \cdot u_1'} u_1'$$

⋮

n. $$u_n' = u_n - \operatorname{Proj}_{u_{n-1}'} u_n - \cdots - \operatorname{Proj}_{u_1'} u_n$$

$$= u_n - \frac{u_n \cdot u_{n-1}'}{u_{n-1}' \cdot u_{n-1}'} u_{n-1}' - \cdots - \frac{u_n \cdot u_1'}{u_1' \cdot u_1'} u_1'$$

Example 4. Let $W = \text{Span}\left\{ \overset{\text{"}u_1}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \overset{\text{"}u_2}{\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}} \right\}$.

basis but <u>not</u> orthogonal.

Compute an orthogonal basis of W.

$$u_1' = u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

these are $\perp$.

$$u_2' = u_2 - \frac{u_2 \cdot u_1'}{u_1' \cdot u_1'} u_1$$

$$= \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} -1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$u_2' = \underbrace{\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{u_2' = u_2 + \frac{1}{2}u_1' \Rightarrow u_2 = u_2' - \frac{1}{2}u_1'} = \begin{bmatrix} -1/2 \\ 7/2 \\ 1/2 \end{bmatrix}$$

Still a basis!

Example 5. Compute an orthogonal basis of Col(A), where

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{cccc} \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ a_1 & a_2 & a_3 & a_4 \end{array} \end{array}$$

① Find some basis of Col(A).

$a_3 = a_1 + a_2$

basis of $A = \{ a_1, a_2, a_4 \}$

② use G-S process.

$$u_1' = a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_2' = a_2 - \frac{a_2 \cdot u_1'}{u_1' \cdot u_1'} u_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_2' = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$$

check $u_1' \cdot u_2' = 0$.

$$u_3' = a_4 - \frac{a_4 \cdot u_1'}{u_1' \cdot u_1'} u_1' - \frac{a_4 \cdot u_2'}{u_2' \cdot u_2'} u_2'$$

$$= \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\frac{5}{\cancel{2}}}{\frac{3}{\cancel{2}}} \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 5/6 \\ -5/6 \\ 0 \\ 5/3 \end{bmatrix}$$

$$u_3' = \begin{bmatrix} 18/6 - 9/6 - 5/6 \\ 0 - 9/6 + 5/6 \\ -2 \\ 3/3 - 5/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -2 \\ -2/3 \end{bmatrix}$$

$u_3' \cdot u_1' = 0$

$u_3' \cdot u_2' = 0$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -2/3 \\ -2 \\ -2/3 \end{bmatrix} \right\}$$