REVIEW OF CH5. & CH6.

Topics:

- Eigenvalues & eigenvectors
- Diagonalization → conditions for diagonalization
  → properties
- Linear transformations
  → Does there exist a basis β s.t. $[T]_\beta$ is diagonal?
- Similarity
- Inner products.
  → standard one
- Projections of vectors → 1-dim subspace
  → >1-dim subspace
- Orthogonal sets & Gram-Schmidt.

Example 1. If possible, diagonalize the matrix A.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 7 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

$$= (2-\lambda)(2-\lambda)(1-\lambda)$$

$$\lambda = 2, 1$$

has multiplicity 2.

$$\lambda = 2: \quad A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 7 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y + 7z = 0, \quad x = 0$$

$y = 7z.$

$$v_1 = \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

Only 1 e-vector associated to $\lambda = 2$

$\Downarrow$

A is not diagonalizable.

Example 2. Determine whether the following statements are true or false.

(A) If M is invertible, then it is also diagonalizable.

FALSE. $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ M is invertible but not diagonalizable.

If M is diagonalizable, then it is invertible if and only if 0 is <u>not</u> an eigenvalue.

$\det(M) = \prod \lambda_i$

(B) Let M be a 4×4 matrix. If M only has 3 distinct eigenvalues, then M is not diagonalizable.

FALSE

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(C) Suppose A & B are both similar to a diagonal matrix D. Then A & B commute.

A & B are similar if $A = PBP^{-1}$.

FALSE.

this could be anything.

$$AB = P_1 D P_1^{-1} P_2 D P_2^{-1} \neq \cdots = BA$$

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix}$$

FACTS!

Diagonalizable $\neq$ Invertible

Let M be a $n \times n$ matrix. If M has n distinct e-value $\underset{\nLeftarrow}{\Rightarrow}$ M is diagonalizable.

Example 3. Let $T: \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by,

$$T(p(x)) = p(1)\cdot x + p'(x).$$

Does there exist a basis β of $\mathbb{P}_2$ such that $[T]_\beta$ is diagonal?

$$[T]_\varepsilon = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Is $[T]_\varepsilon$ diagonalizable?

$$\det([T]_\varepsilon - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 3 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = -\lambda(\underbrace{\lambda^2-\lambda-1}_{\text{factor}})$$

$$\lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

$\lambda = 0, \lambda_+, \lambda_-$; 3 distict e-values

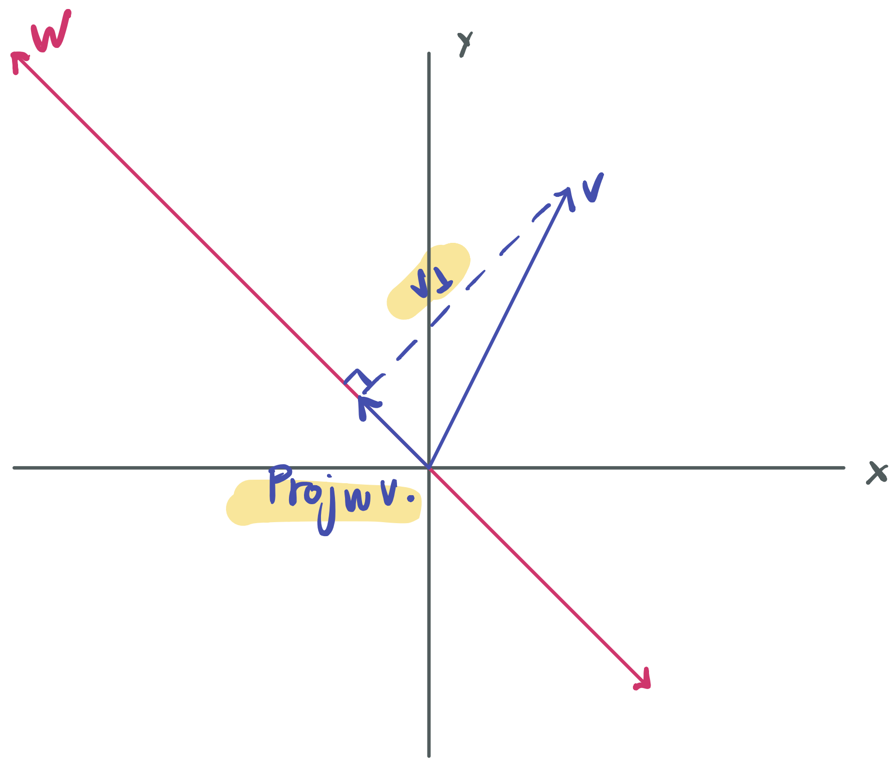
$\Rightarrow [T]_{\mathcal{E}}$ is diagonalizable.

$\Rightarrow \exists$ a basis β s.t. $[T]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_+ & 0 \\ 0 & 0 & \lambda_- \end{bmatrix}$

Example 4. Compute the projection of the vector $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ onto the subspace $\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = W$.

$$\underset{u}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$\text{Proj}_W v = \frac{v \cdot u}{u \cdot u} u = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$



Example 5. Let $W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$. Compute a basis for the subspace $W^\perp$. Show that for any $v \in \mathbb{R}^3$, $v = w + w'$ where $w \in W$ and $w' \in W^\perp$.

$w' \in W^\perp$, $\quad w' \cdot w = 0 \;\; \forall w \in W$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot c \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\cancel{c} \cdot (2y + z) = 0$$

$$z = -2y$$

x can be anything.

$$w' = \begin{bmatrix} x \\ y \\ -2y \end{bmatrix} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

Claim: $\left\{ \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{=u_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{=u_2}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}_{=u_3} \right\}$

is a basis of $\mathbb{R}^3$.

$v \in \mathbb{R}^3$, $v = c_1 u_1 + \underbrace{c_2 u_2 + c_3 u_3}$

$v = w + w'$.

$w \in W$ $\quad w' \in W^\perp$.

Example 6. Give an orthogonal basis for the subspace $W^\perp$ where $W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Example 7. Compute an orthogonal basis of Col(A),

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}$$

Gram-Schmidt!

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} - \frac{5}{6} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 10/3 \\ 5/6 \\ 5/6 \end{bmatrix}$$

$$= \begin{bmatrix} -10/3 \\ 1/6 \\ 19/6 \end{bmatrix}$$

$u_1 \cdot u_2 = 0$ ✓

Let $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ what is $\text{Proj}_{\text{Col}(A)} v$?

MUST have $\perp$ basis of col(A).

$$\text{Proj}_{\text{col}(A)} v = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2$$