Nonhomogeneous Linear Equations

$$ay'' + by' + cy = F(x) \quad (*)$$

$$ay'' + by' + cy = 0 \quad (**)$$

$(*)$ is a nonhomogeneous, second order DE with constant coefficients.

RHS = $F(x)$.

The solution to $(*)$ is closely related to the solution of $(**)$ so let's start with an example.

Ex 1. $y'' - 2y' - 3y = 0$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0 \Rightarrow r_1 = 3, \ r_2 = -1$$

$$\boxed{y = c_1 e^{3x} + c_2 e^{-x}}$$

<u>Thm</u>. Let y_c be the most general solution of (**), then the solution of (*) has the form

$$y(x) = y_c(x) + y_p(x)$$

where y_p is some <u>particular solution</u> of (*). The above is the general solution to (*).

<u>Ex</u> 2. Consider the differential equation,

$$y'' - 2y' - 3y = e^x$$

Show that $y_p(x) = -\frac{1}{4}e^x$ is a particular solution. Compute the general solution of the above as well.

$$y_p = -\frac{1}{4}e^x \qquad y_p' = -\frac{1}{4}e^x \qquad y_p'' = \frac{-1}{4}e^x$$

$$\begin{aligned} y_p'' - 2y_p' - 3y_p &= \frac{-1}{4}e^x - 2\left(\frac{-1}{4}e^x\right) - 3\left(\frac{-1}{4}e^x\right) \\ &= -\frac{1}{4}e^x + \frac{2}{4}e^x + \frac{3}{4}e^x \\ &= \left(-\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right)e^x = e^x \end{aligned}$$

Now to compute the general solution, let's compute y_c, which is the general solution of,

$$y'' - 2y' - 3y = 0$$

From example 1 we know that,

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

So the general solution to $y'' - 2y' - 3y = e^x$ is,

$$y = y_p + y_c$$

$$\boxed{y = -\frac{1}{4} e^x + c_1 e^{3x} + c_2 e^{-x}}$$

So now our goal is to learn methods for solving for y_p. There are two ways to go about this:

1. Method of undetermined coefficients
2. Method of variation of parameters.

Method of undetermined coefficients.

$F(x) =$ polynomial.

Guess at what $y_p(x)$ is. Our guess will be a polynomial that is the same degree of $F(x)$.

↳ because if y is a polynomial of deg n then so is $ay'' + by' + cy$.

Ex3. Solve for the particular solution to

$$y'' - y = x^2 + x$$

We see that $F(x) = x^2 + x$ is a polynomial of degree 2. So we make the guess,

$$y_p = \underline{A}x^2 + \underline{B}x + \underline{C}$$

Also a polynomial of deg=2.

undetermined coefficients we will determine.

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Want to assign A, B, C so that this is true.

$$y_p'' - y_p = 2A - Ax^2 - Bx - C$$

$$= -Ax^2 - Bx + (2A - C) = x^2 + x$$

$$-A = 1 \Rightarrow A = -1$$

$$-B = 1 \Rightarrow B = -1$$

$$2A - C = 0 \Rightarrow -2 - C = 0$$

$$C = -2$$

So we see that

$$y_p = -x^2 - x - 2 \quad \text{(the particular solution).}$$

<u>Exercise</u>. Solve for the general solution to

$$y'' - y = x^2 + x.$$

<u>Answer</u>. $y = c_1 e^{x} + c_2 e^{-x} - x^2 - x - 2$.

<u>$F(x)$ = exponential.</u>

<u>Ex 4.</u> Solve for the particular solution to

$$y'' + y' = e^{2x}$$

Since $F(x)$ is an exponential, we should guess an exponential (w/ the same exponent).

$$y_p = \underline{A}\, e^{2x}$$

undetermined coefficient.

$$y_p = Ae^{2x} \qquad y_p'' + y_p' = 4Ae^{2x} + 2Ae^{2x} = 6Ae^{2x} = e^{2x}$$

$$y_p' = 2Ae^{2x} \qquad \Rightarrow 6A = 1$$

$$y_p'' = 4Ae^{2x} \qquad \Rightarrow A = \frac{1}{6}$$

So the particular solution is, $y_p = \frac{1}{6}e^{2x}$.

<u>Exercise</u>. Compute the general solution of

$$y'' + y' = e^{2x}$$

<u>Answer</u>. $y = c_1 e^{-x} + c_2 + \frac{1}{6}e^{2x}$.

Try the following: (*) do at end.

1. $y'' + 4y = 2(x+1)$

   $y = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{2}x + \frac{1}{2}$

2. $y'' + 5y' + 6y = 18x + 6e^x$

   $y = c_1e^{-3x} + c_2e^{-2x} + 3x - \frac{5}{2} + \frac{1}{2}e^x$.

3. $y'' - 2y' = e^{2x}$

   $y = c_1e^{2x} + c_2 + \frac{1}{2}xe^{2x}$

Let's summarize what we've determined so far,

Method of undetermined coefficients:

(*) guess the ~~value of~~ form of y_p and plug it ~~in~~ into the differential equation to determine the coefficients.

(1) $F(x) = \underline{a_n}x^n + \underline{a_{n-1}}x^{n-1} + \dots + \underline{a_1}x^1 + \underline{a_0}$

known - given in the problem.

$y_p(x) = \underline{A}x^n + \underline{B}x^{n-1} + \dots + \underline{C}x + \underline{D}$

unknown - what we solve for.

(2) $F(x) = ae^{kx} \rightsquigarrow y_p(x) = \underbrace{A}_{\text{unknown.}} e^{kx}$

F(x) = exponential. more.

There are some exceptions to the rule above, let's see one.

Ex5. Compute the particular solution to

$$y'' - y = e^x$$

Base on what we've discussed, we should guess

$$y_p = Ae^x$$

$y_p' = Ae^x$

$y_p'' = Ae^x$

$$y_p'' - y_p = Ae^x - Ae^x = 0 \neq e^x$$

can't happen.

What happened for this to fail? Well $F(x) = e^x$ is a solution to the corresponding homogeneous problem, $y'' - y = 0$.

If this happens we should instead guess,

$$y_p = Axe^x$$

multiply by x.

$y_p' = Ae^x + Axe^x$

$y_p'' = 2Ae^x + Axe^x$

$$y_p'' - y_p = 2Ae^x + Axe^x - Axe^x$$

$$= 2Ae^x = e^x$$

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

So the particular solution of $y''-y=e^{x}$ is

$$\boxed{y_p = \frac{1}{2}xe^{x}}$$

Computing the general solution of (*):

1. Compute the general solution of (**)
2. Find a particular solution

<u>Ex6.</u> Find the general solution of

$$y''+3y'+2y=6x^2$$

1. Find the general solution to $y''+3y'+2y=0$.

$r^2+3r+2=(r+2)(r+1)=0$, $r_1=-2$, $r_2=-1$.

$y_c=c_1e^{-2x}+c_2e^{-x}$

2. Find particular solution:

$y_p = Ax^2+Bx+C$

$$y_p' = 2Ax + B \qquad y_p'' + 3y_p' + 2y_p = 2A + 3(2Ax + B)$$

$$y_p'' = 2A \qquad\qquad + 2(Ax^2 + Bx + C)$$

$$= 2Ax^2 + (6A + \overset{2}{\cancel{3}}B)x + (2A + 3B + 2C)$$

$$2Ax^2 + (6A + \overset{2}{\cancel{3}}B)x + (2A + 3B + 2C) = 6x^2$$

$$2A = 6 \Rightarrow A = 3$$

$$6A + 2B = 0$$

$$18 + 2B = 0 \Rightarrow B = -9$$

$$2A + 3B + 2C = 0$$

$$6 - 27 + 2C = 0 \Rightarrow C = \frac{21}{2}$$

So $y_p = 3x^2 - 9x + \frac{21}{2}$ and the general solution of $y'' + 3y' + 2y = x^2$ is,

$$\boxed{y = c_1 e^{-2x} + c_2 e^{-x} + 3x^2 - 9x + \frac{21}{2}}$$

Find the general solution of the following.

1. $y'' + 9y = 3e^{3x}$ $\left(y = c_2 \sin(3x) + c_1 \cos(3x) + \frac{1}{6}e^{3x}\right)$

2. $y'' + 2y' + y = x$ $\left(y = c_1 e^{-x} + c_2 x e^{-x} + x - 2\right)$

<u>$F(x) =$ sin or cos.</u>

Our guess for y_p should be a linear combination of sin's <u>and</u> cos's.

even if only one is in $F(x)$.

<u>Ex 7.</u> $y'' + 2y' - 3y = \sin(x)$ (Find the particular solution).

$y_p = A\sin(x) + B\cos(x)$

$y_p' = A\cos(x) - B\sin(x)$

$y_p'' = -A\sin(x) - B\cos(x).$

$$-A\sin x - B\cos x + 2(A\cos x - B\sin x) - 3(A\sin x + B\cos x)$$

$$= (-A - 2B - 3A)\sin x + (-B + 2A - 3B)\cos x$$

$$= (-4A - 2B)\sin x + (-4B + 2A)\cos x = \sin x$$

$$\left.\begin{array}{r} -4A - 2B = 1 \\ 2 \times (+2A - 4B = 0) \end{array}\right\} \Rightarrow \begin{array}{r} -4A - 2B = 1 \\ 4A - 8B = 0 \\ \hline -10B = 1 \end{array}$$

$$B = -\frac{1}{10}$$

$$2A - 4\left(-\frac{1}{10}\right) = 0$$

$$2A + \frac{2}{5} = 0 \Rightarrow A = -\frac{1}{5}$$

$$\boxed{y_p = -\frac{1}{10}\cos x - \frac{1}{5}\sin x}$$

Notice that even though $F(x)$ just had $\sin x$, y_p involves both $\sin x$ and $\cos x$.

<u>Exercise</u>. Find the particular solution to,

$$y'' + 2y' = 4\cos(2x)$$

<u>Answer</u>. $\cancel{y''+2y'=}$ $y_p = \frac{1}{2}\sin(2x) - \frac{1}{2}\cos(2x)$.

F(x) = combination of things.

Ex 8. Find the particular solution to,

$$y'' + 2y = 3xe^x$$

polynomial * exponential.

Guess: $y_p = (\underline{A}x + \underline{B}) * e^x$ ← no need to put a coefficient here

polynomial * exponential.

$$y_p = Axe^x + Be^x$$

$$y_p' = Ae^x + Axe^x + Be^x = Axe^x + (A+B)e^x$$

$$y_p'' = Axe^x + (2A+B)e^x$$

$$y_p'' + 2y_p = Axe^x + (2A+B)e^x + 2(Axe^x + Be^x)$$

$$= 3Axe^x + (2A+3B)e^x = 3xe^x$$

$$3A = 3 \Rightarrow A = 1$$

$$2A + 3B = 0 \Rightarrow 2 + 3B = 0 \Rightarrow B = -\frac{2}{3}$$

$$\boxed{y_p = xe^{x} - \frac{2}{3}e^{x}}$$

Ex 9. Find the particular solution to

$$y'' + 4y' + 4y = x + e^{-x}$$

polynomial + exponential.

$$y_p = Ax + B + Ce^{-x}$$

$$y_p' = A - Ce^{-x}$$

$$y_p'' = Ce^{-x}$$

$$Ce^{-x} + 4(A - Ce^{-x}) + 4(Ax + B + Ce^{-x})$$

$$= 4Ax + (4A + 4B) + (C \cancel{- 4C} + \cancel{4C})e^{-x} = x + e^{-x}$$

$$4A = 1 \Rightarrow A = \frac{1}{4}$$

$$4A + 4B = 0 \Rightarrow B = -\frac{1}{4}$$

$$C = 1$$

$$\boxed{y_p = \frac{1}{4}x - \frac{1}{4} + e^{-x}}$$

Again we can solve initial value problems & boundary value problems concerning these equations.

Ex 10. $y'' + y = x$, $y(0) = 1$, $y(\frac{\pi}{2}) = \pi$.

1. Solve for the general solution of $y'' + y = 0$.

$r^2 + 1 = 0 \Rightarrow r = \pm i$

$$\boxed{y_c = c_1 \cos x + c_2 \sin x}$$

2. Solve for the particular solution to

$y'' + y = x$

$y_p = Ax + B$

$y_p'' = 0$

$$\boxed{y_p = x}$$

$y_p'' + y_p = Ax + B = x$

$A = 1$

$B = 0.$

3. Solve the boundary value problems,

$$y = c_1 \cos x + c_2 \sin x + x$$

$$y(0) = 1 = c_1 \cos(0) + c_2 \sin(0) + 0$$

$$1 = c_1$$

$$y = \cos x + c_2 \sin x + x$$

$$y\left(\frac{\pi}{2}\right) = \pi = \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$\pi = c_2 \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$\frac{\pi}{2} = c_2$$

$$\boxed{y = \cos x + \frac{\pi}{2} \sin x + x}$$

Exercises.

1. $y'' + y = \sin(2x)$

2. $y'' + 2y' + y = x + 3e^{2x}$

3. $y'' + 4y' + 4y = x^2$, $y(0) = \frac{3}{8}$, $y'(0) = \frac{1}{2}$

Answers.

1. $y = c_1 \sin x + c_2 \cos x - \frac{1}{3}\sin(2x)$.

2. $y = c_1 e^{-x} + c_2 x e^{-x} + x - 2 + \frac{1}{3}e^{2x}$

3. $y = xe^{-2x} + \frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{8}$