Now what if our second order differential equation has the form

$$ay'' + by' + cy = F(x) \tag{1}$$

where $F(x)$ is a continuous functions, often called a forcing term (in reference to physics). To solve this we consider the complimentary homogeneous equation,

$$ay'' + by' + cy = 0 \tag{2}$$

Theorem. The general solution to the nonhomogeneous differential equation (1) can be written as

$$y(x) = y_H(x) + y_p(x)$$

where y_p is a particular solution to (1) and y_H is the general solution to the complimentary equation (2).

Proof.

We know how to solve for y_H, so the question becomes how to find y_p. I will first introduce a systematic method called **variation of parameter** and then turn to a faster (but not deterministic) method called **undetermined coefficients**.

The Method of Variation of Parameter

Suppose we have already solved for the homogeneous solution and it has the form

$$y_H(x) = c_1 y_1(x) + c_2 y_2(x)$$

To solve for the particular solution we suppose it has the form

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

From here we wish to compute u_1 and u_2, and to do so we place two constraints on the functions (two functions = two constraints). The first constraint is that y_p does in fact solve our differential equation (1) and the second is one of our choosing,

$$u_1'y_1 + u_2'y_2 = 0$$

We choose this as it will simplify our calculations. Let's use this to derive a mathematical equation for the first constraint,

In summary we solve for u_1 and u_2 by solving the two equations,

$$u_1'y_1 + u_2'y_2 = 0 \tag{3}$$

$$a\left(u_1'y_1' + u_2'y_2'\right) = F \tag{4}$$

EXAMPLE 1. Solve the equation $y'' + y' - 2y = x^2$

EXAMPLE 2. Solve the equation $y'' + 4y = e^{3x}$

EXAMPLE 3. Solve the equation $y'' + y = \tan x$