Lec 33: Second Order Linear Equations, cont'd (17.1)

Recall from last time the three cases we may encounter when solving second order homogenous differential equation with constant coefficients

$$ay'' + by' + cy = 0 \tag{1}$$

1. Characteristic equation has two distinct roots.
2. Characteristic equation has one repeated (real) root.
3. Characteristic equation has two complex conjugate roots.

Last time we dealt with the first case, so now let us consider case 2, one repeated (real) root. If the characterstic equation has one real root r, then the general solution of (1) is,

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}$$

First let's show that $y_1(x) = e^{rx}$ and $y_2(x) = xe^{rx}$ are linearly independent solutions.

Now let's show that $c_1 e^{rx} + c_2 x e^{rx}$ satisfies (1),

EXAMPLE 1. Solve the equation $4y'' + 12y' + 9y = 0$.

Now let's consider the third and final case, where there are two complex conjugate roots of the form, $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. These are technically distinct roots, so lets treat them as if they were case 1, and just be careful about the complex exponential we have floating around.

From this we see that if the roots of the characteristic equation are complex numbers $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ then the general solution to (1) is

$$y(x) = e^{\alpha x} \left(c_1 \cos(\beta x) + \sin(\beta x) \right)$$

We may skip to this form of the solution every single time we encounter complex roots.

EXAMPLE 2. Solve the equation $y'' - 6y' + 13y = 0$.

With first order differential equations we were introduced to initial value problems, now for second order differential equations we have initial value problems (IVPs) and boundary value problems (BVPs).

EXAMPLE 3. Solve the IVP

$$y'' + y = 0 \; y(0) = 2 \; y'(0) = 3$$

EXAMPLE 4. Solve the BVP

$$y'' + 2y' + y = 0 \; y(0) = 1 \; y(1) = 3$$