We will end our discussion on section 9.5 by discussing **Bernoulli differential equations** which have the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1}$$

See that if $n = 0$ or 1, then the above equation is linear. Else we can use a change of variable to transform it into a first order differential equation.

EXAMPLE 1. Use the substitution $u = y^{1-n}$ to transform equation (1) into a linear equation.

EXAMPLE 2. Solve $xy' + y = -xy^2$.

We will now begin our discussion on second order differential equations (which is the last major topic we will learn!).

Definition. A **second order linear differential equation** has the form,

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x) \tag{2}$$

If $G(x) = 0$ we say that the differential equation is **homogeneous**. Until section 17.4 we will only deal with $P(x)$, $Q(x)$, and $R(x)$ as constant functions. And for this section, 17.1, we will only deal with $G(x) = 0$. Thus we are considering differential equations of the form,

$$ay'' + by' + cy = 0 \tag{3}$$

Theorem. If $y_1(x)$ and $y_2(x)$ are both solutions to equation (2) with $G(x) = 0$ and c_1 and c_2 are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of equation (2) (with $G(x) = 0$).

Proof.

In order to find the *general solution* of a second order differential equation, we must find two **linearly independent** solutions.

Definition. Two functions $f(x)$ and $g(x)$ are linearly independent if $c_1f(x) + c_2g(x) = 0$ if and only if $c_1 = c_2 = 0$. In other words, f and g are not constant multiples of each other.

EXAMPLE 3. Are $f(x) = x^2$ and $g(x) = 5x^2$ linearly independent? What about $f(x) = e^x$ and $g(x) = e^{2x}$?

Theorem. If y_1 and y_2 are linearly independent solutions of equation (2) on an interval and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Now we will begin to consider specific cases of equation (3) (where $a \neq 0$. Let's guess at possible solutions to this equation and try $y = e^{rx}$ where r is some unknown constant. Substituting this into (3) we get,

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0 \tag{4}$$

$$\left(ar^2 + br + c\right) e^{rx} = 0 \tag{5}$$

The last statement implies the condition

$$ar^2 + br + c = 0 \tag{6}$$

Which we refer to as the **characteristic (or auxiliary) equation**. Notice that this comes from the replacement y'' $(y^{(2)}) \to r^2$, y' $(y^{(1)}) \to r$, and y $(y^{(0)}) \to 1$.

The characteristic equation is just a quadratic polynomial and thus we can solve for the roots. We wish to consider three possible cases for the roots:

1. Two distinct roots $\left(b^2 - 4ac > 0\right)$
2. One repeated root $\left(b^2 - 4ac = 0\right)$
3. Two complex conjugate roots $\left(b^2 - 4ac < 0\right)$

Today we will only consider case 1, two distinct roots. In this case,

$$ar^2 + br + c = 0$$

has two distinct solutions we will call r_1 and r_2. In this case the differential equation (3) has the general solution

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Let us prove this solves (3),

EXAMPLE 3. Find the general solution to the differential equation $y'' + y' - 6y = 0$.