Definition. A first order differential equation is called **linear** if it can be put in the form,

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

In some special cases of $P(x)$ and $Q(x)$, equation (1) is separable, but in most cases it is not. There is a general method for solving equation (1) which we will call the **integrating factor method**. Let's first consider the following examples, which we will use the product rule for derivatives to help us solve.

EXAMPLE 1. Solve the first order linear differential equation $xy' + y = 2x$.

EXAMPLE 2. Solve the first order linear differential equation $x^2\frac{dy}{dx} + 2xy = e^x$.

The above two examples illustrate the method we will use to solve the general case of equation (1). We wish to find some function, call it $I(x)$, such that multiplying equation (1) by $I(x)$ gives us a product rule on the left hand side.

Definition. The i**integrating factor** of equation (1) is

$$I(x) = e^{\int P(x)\, dx} \tag{2}$$

Note that the first order differential equation must be exactly in the form of equation (1) for the integrating factor to be correct.

Why does this choice of $I(x)$ work?

To solve the linear differential equation $y' + P(x)y = Q(x)$ multiply both sides by the integrating factor $I(x) = e^{\int P(x)\, dx}$ and integrate both sides.

$$y' + P(x)y = Q(x) \tag{3}$$
$$I(x)y' + I(x)P(x)y = I(x)Q(x) \tag{4}$$
$$I(x)y' + I'(x)y = I(x)Q(x) \tag{5}$$
$$(I(x)y)' = I(x)Q(x) \tag{6}$$
$$I(x)y = \int I(x)Q(x)\, dx \tag{7}$$
$$y = \frac{1}{I(x)} \int I(x)Q(x)\, dx \tag{8}$$

EXAMPLE 3. Solve $y' + 3xy = 6x$.

EXAMPLE 4. Solve the initial value problem $x^2y' + xy = 1$, $x > 0$, $y(1) = 2$.