Lec 30: Separable Equations, cont'd (9.3) and Models for Population Growth (9.4)

In this lecture we will continue to look at first order differential equations that are separable, however we will turn our focus to models that are often used to represent population growth. Let's start with a model that was introduced yesterday, the exponential growth model

$$\frac{dP}{dt} = kP \tag{1}$$

where k is some constant. What does this equation mean as a population model? dP/dt is the **rate of population growth**, this could be thought of as a birth rate or as a net rate, birth rate - death rate. The left hand side of (1) can be read as "proportional to the population". Thus this model tells us,

The rate of population growth is proportional to the population.

If we let $k > 0$ then the rate of population growth increases as the population gets larger, and if $k < 0$ the rate of growth is negative and gets more negative as the population size gets larger. Last lecture we showed that (1) has the family of solutions,

$$P(t) = Ce^{kt} \tag{2}$$

And see when $t = 0$ the equation becomes, $P(0) = C$, thus C is thought of as the initial population size. Let's graph the results for $k > 0$ and $k < 0$ given some initial population size P_0.

What's realistic and unrealistic about this model? It is a good guess that the population growth rate is proportional to the population when it is relatively small, but as the population grows we imagine resources will play a roll in how big the population can get. More animals (plants, etc) means more competition for resources (such as food and shelter) and greater chances for the spread of disease. A model that does a slightly better job at accounting for these is called the **logistic model of population growth** which has the form,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \tag{3}$$

Before we solve this, let's look at the qualitative properties of this differential equation.

- When P is small, $(1 - P/M) \approx 1$ and so $dP/dt \approx kP$. In other words, when P is small the logistic growth model behaves like exponential growth.
- When P is close to, but less than M, then $dP/dt \approx \epsilon$, where $\epsilon > 0$ and $\epsilon \to 0$ as $P \to M^-$.
- If P is greater than M then $dP/dt < 0$ and the population is decreasing in size.

Let's graph P vs. dP/dt and draw the "phase diagram" for t vs. P

Now let us solve for the actual function that solves equation (2).

EXAMPLE 1. Write the solution to the initial value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \quad P(0) = 100$$

Other models of populations in one variable include the harvesting model,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

and the edited logistic which includes a minimum population,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$