Lec 24: Representations of Functions as Power Series (11.9)

Recall in the last section we said a power series represented a function, $f(x)$ whenever x was in the radius of convergence. What kind of functions can we represent? We can represent any function as a power series (though some are more difficult than others), we can even use series to express functions that we couldn't express with elementary functions. Let's start with a power series we know how to derive already... recall from section 11.2,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \tag{1}$$

whenever $|r| < 1$. Let $a = 1$ and replace r with x and we get,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{2}$$

We have now found a power series for the function $f(x) = 1/(1-x)$ (about $x = 0$), which has a radius of convergence $|x| < 1$. From this we are able to find the power series representation for a whole slue of functions.

EXAMPLE 1. Find a power series representation of the function $f(x) = \dfrac{1}{1-3x}$, and then compute the radius of convergence of the series.

EXAMPLE 2. Find a power series representation of the function $f(x) = \dfrac{x}{1-x^2}$.

EXAMPLE 3. Find a power series representation of the function $f(x) = \dfrac{1}{2-x}$.

From the above we have seen that we can do normal algebraic operations to power series, but what about differentiation and integration? For that we have the following theorem...

Theorem. If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{3}$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \tag{4}$$

$$\int f(x)\,dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} \tag{5}$$

The radii of convergence of the power series in (4) and (5) is also R (though the interval of convergence may not be the same).

In words the above theorem says we can differentiate (integrate) a function by differentiating (integrating) each term in the power series. In doing so we get a power series of the derivative (integral). This is often called **term by term differentiation (integration).**

EXAMPLE 4. Find the power series of the function $f(x) = \dfrac{1}{(1-x)^2}$.

EXAMPLE 5. Find the power series of the function $f(x) = \ln(1+x)$.

EXAMPLE 6. Find the power series of the function $f(x) = \dfrac{x}{(1+4x)^2}$.

EXAMPLE 7. Find the power series of the function $f(x) = \int_0^x \frac{t}{1+t^8}\, dt$.