Lec 23: Power Series (11.8)

Definition. A series of the form,

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{1}$$

is called a **power series about** a, where the c_n's are the coefficients of the series.When the power series converges, the series represents a function, $f(x)$.

EXAMPLE 1. For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n+1}$ converge.

EXAMPLE 2. For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n!}x^n$ converge.

Theorem. For a given power series $\sum c_n(x-a)^n$, there are only 3 possibilities (regarding convergence:

1. The series converges absolutely only when $x = a$.
2. The series converges absolutely for all x.

3. There exists a positive (finite) number R such that the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$. The series my converge absolutely, converge conditionally, or diverge when $|x-a| = R$.

Definition. The number R in case (3) above is called the **radius of convergence** of the power series. In case (1) we say $R = 0$ and in case (2) we say $R = \infty$.

To find R we will always use the ratio (or on rare occasions root) test, just like we did in examples (1) and (2). Then we are usually interested if the series converges for $|x-a| = R$, so we check these "endpoints" individually. The radius and the endpoints together give us,

Definition. The **interval** of convergence is all x for which the series converges (note that it will always be an interval or the single point $x = a$ when $R = 0$).

EXAMPLE 3. Find the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{3^n n}$ and discuss the convergence at the endpoints.

EXAMPLE 4. Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n^2}$

EXAMPLE 5. Find the interval and radius of convergence of $\sum_{n=1}^{\infty} \frac{(nx)^n}{n!}$