Lec 20: Absolute Convergence and the Ratio and Root Tests (11.6)

We notice something curious in the last section and that was that the series $\sum \frac{1}{n}$ diverges, but the series $\sum \frac{(-1)^n}{n}$ converges (by the alternating series test). Alternatively note that $\sum \frac{1}{n^2}$ and $\sum \frac{(-1)^n}{n^2}$ both converge. So can we categorize the difference between the alternating harmonic, $\sum \frac{(-1)^n}{n}$, and $\sum \frac{(-1)^n}{n^2}$? We do, in fact, have vocabulary for this, which we will cover in this lecture.

Definition. A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values, $\sum |a_n|$ is convergent.

Definition. A series $\sum a_n$ is called **conditionally convergent** if it is convergent, but not absolutely convergent.

Theorem. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

We will prove the above theorem in a slightly different manner, using something called the **triangle inequality**. You have probably been introduced to this inequality geometrically; given three sides of a triangle, a_1, a_2, and a_3, $a_i + a_j \geq a_k$, $i \neq j \neq k$ and $i,\ j,\ k = 1,\ 2,\ 3$. We will introduce a more general version without proof,

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \tag{1}$$

this inequality can also be generalized to the infinite case (think an infinite sum).

Proof.

EXAMPLE 1. Classify the convergence of the following series,

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

If a series consists of only positive terms, then convergence is equivalent too absolute convergence. In other words, a series can only converge conditionally if it has infinitely many positive and negative terms (we will only deal with alternating series). For the remainder of this lecture we will introduce two tests for absolute convergence.

The Ratio Test.

(i) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then the series $\sum a_n$ is absolutely convergent.

(ii) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then the series $\sum a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$ the ratio test is inconclusive.

Note that the ratio test proves absolute convergence, but can prove ANY series diverges (including alternating series). However there are some series that converge and diverge that the ratio test fails to give us a result for.

EXAMPLE 2. Use the ratio test on the following series and state the result, if the ratio test is inconclusive determine the convergence or divergence of the series using another method.

- $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

- $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{2^n}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

- $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$

Root Test.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then the series $\sum a_n$ is absolutely convergent.
(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ then the series $\sum a_n$ is divergent.
(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$ the root test is inconclusive.

The root test, in general, will not be as useful as the ratio test, but works well for some (otherwise tricky) problems.

EXAMPLE 3. Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$