Up until this point we have only dealt with series that have positive terms, but what happens when a series has negative terms? We will begin to learn about these series, starting with what we call an alternating series.

Definition. An **alternating series** is a series whose terms are alternately positive and negative. Thus the series has infinitely many positive and negative terms.

When a series is alternating, the criteria for convergent are more "relaxed" in the sense that the terms of the series can go to zero slower than what we previously thought, so long as they are alternating positive and negative. This is revealed in the following theorem,

Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 + \cdots \quad b_n > 0 \tag{1}$$

satisfies

$$b_{n+1} \leq b_n \text{ for all } n \tag{2}$$
$$\lim_{n\to\infty} b_n = 0 \tag{3}$$

then the series is convergent.

EXAMPLE 1. Show that the alternating harmonic series converges.

EXAMPLE 2. Does the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{n^3+1}$ converge or diverge?

Sometimes it's less obvious that a series is even alternating, so it is good practice to write out the first couple of terms,

EXAMPLE 3. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)\sin\left(\frac{1}{n}\right)$ converges or diverges.

For alternating series we also have a useful estimation theorem,

Alternating Series Estimation Theorem. If $s = \sum(-1)^{k+1}b_k$, where $b_k > 0$, is the sum of an alternating series that satisfies

$$b_{k+1} \leq b_k \tag{4}$$
$$\lim_{k\to\infty} b_k = 0 \tag{5}$$

then

$$|R_n| = |s - s_n| \leq b_{n+1} \tag{6}$$

Note that the alternating series needs to converge in order for us to use this theorem and that the series MUST be alternating. A proof of the theorem is given in the text.

EXAMPLE 4. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ correct to 3 decimal places.