Sometimes we wish to compare one series to another using the direct comparison test, however it isn't always easy to get the inequality we want, for example

EXAMPLE 1(a). Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2 - 3n}$ converge or diverge?

We know that the above converges, however using the direct comparison test was a bit tedious. The "feeling" is that the n term becomes irrelevant after a while and the sequence decreases "like" $1/n^2$. But is there a way we can prove this? Yes, but we need the limit comparison test.

The Limit Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \tag{1}$$

where c is a finite number that is not 0, then either both series converge or both diverge.

When applying the limit comparison test in general we compare a series we do not know about to a series that we know the convergence/divergence of (or at least to a simpler series).

EXAMPLE 1(b). Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2 - 3n}$ converges or diverges.

EXAMPLE 2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges or diverges by comparing the series to (i) $\sum_{n=1}^{\infty} \frac{1}{3^n}$ and then (ii) $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

Note in the first case of example (2) we got $c = 0$ which is an **inconclusive** result for the limit comparison test. We must take extra care in picking the right series to compare with, so we get a non-zero, non-infinite limit. A zero or infinite limit **does not** mean we chose the wrong convergence/divergence, it means we need to reevaluate and try again.

EXAMPLE 3. Determine whether the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

EXAMPLE 4. Determine whether the series $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$ converges or diverges.