When we previously looked at the harmonic series we compared it with the integral of its corresponding function,

$$\int_1^\infty \frac{1}{x}\,dx \tag{1}$$

We noted that both diverged (and that this was probably connected). Now let's consider a convergent improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx \tag{2}$$

And let's use this to geometrically convince ourselves that the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges as well.

To show the divergent case we use a similar argument, but instead consider the rectangles whose tops lie above the curves. The formal proof of the integral test is given at the end of section 11.3, and I suggest you look at it. Let us state the integral test,

Theorem (Integral Test). Suppose f is a *continuous, positive, decreasing* function on $[1,\infty)$ and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent.

EXAMPLE 1. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge?

EXAMPLE 2. For what values of p does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

This leads us to the p-test for series...

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent otherwise.

EXAMPLE 3. Does the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converge?

Unfortunately, the integral test **does not** tell us the value of the sum, only whether or not it exists. BUT, notice from our original graph that as the integral persists, the rectangular sums become closer to the exact area of the integral. Thus we can use the tail of the integral to get a decent estimate on the remainder of the sum.

Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_k$ is convergent. Let $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx \tag{3}$$

and from this we get the following estimate of the sum,

$$s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\, dx \tag{4}$$

EXAMPLE 4. Estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ using $n = 3$.