A **series** is a sum of infinitely many terms which we can denote,

$$a_1 + a_2 + a_3 + \cdots = \sum_{i=1}^{\infty} a_i \tag{1}$$

and the question we wish to ask is, *does it make sense to talk about the sum of infinitely many terms?*

It will help first if we can translate this infinite sum into an infinite sequence, something we have an easier time understanding. In order to do so, we define the **sequence of partial sums**. A partial sum is just a truncated version of the series in (1).

$$s_1 = a_1 \tag{2}$$

$$s_2 = a_1 + a_2 \tag{3}$$

$$s_3 = a_1 + a_2 + a_3 \tag{4}$$

$$s_n = a_1 + a_2 + \cdots + a_N = \sum_{i=1}^{n} a_i \tag{5}$$

Definition. Given a series $\sum_{i=1}^{\infty} a_i$ we call s_n in (5) the nth partial sum. If the sequence $\{s_n\}$ converges to a finite limit, $\lim_{n\to\infty} s_n = s$, then the series is **convergent** and we write

$$\sum_{i=1}^{\infty} a_i = s \tag{6}$$

Where s is the **sum** of the series. If $\{s_n\}$ diverges, then the series is **divergent**.

EXAMPLE 1 Calculate the sum of the geometric series $\sum_{i=1}^{\infty} ar^{i-1}$ and comment on which r the sum exists for.

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Since the above was for any general geometric series, this is the formulation we can use every time (and skip the details), geometric series are one of the few that we can easily calculate the sum of. BE CAREFUL, however, of indices which do not match the formula - in these cases a little algebra is necessary.

EXAMPLE 2. Is the series $\sum_{i=0}^{\infty} 2^n 5^{1-n}$ convergent or divergent, and if it converges what is the sum?

EXAMPLE 3. What does the series $\sum_{i=0}^{\infty} x^i$ converge to for $|x| < 1$

This is our first example of a **Taylor series**, but you don't have to worry about that is right now.

There is another type of series problem in which we can extract the sum using partial sums. These problems are called **telescoping** problems,

EXAMPLE 4. Find the sum of the series $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$

One of the most important series you'll learn about is called the **harmonic series** and written as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \tag{7}$$

The harmonic series is DIVERGENT and the proof is sketched in the book. It is useful to relate this to the integral we considered back in 7.8,

$$\int_1^{\infty} \frac{1}{x}\, dx \tag{8}$$

Just like the harmonic series this integral is divergent and if you think the two are related, you are right.

Theorem (Test for Divergence). If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

We can also conclude the *contrapositive* from this theorem, that is, if $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$. BE CAREFUL though, this theorem does **not** tell us that if $\lim\limits_{n\to\infty} a_n = 0$

then the series $\sum_{n=1}^{\infty} a_n$ converges. Again,

$$\lim_{n\to\infty} a_n = 0 \;\not\Longrightarrow\; \sum_{n=1}^{\infty} a_n \text{ converges} \tag{9}$$

A good way to remember this is to think of the harmonic series, as $\lim_{n\to\infty} \frac{1}{n} = 0$ however we know the series diverges. So if we try to apply the above theorem and calculate the limit to be 0, we cannot say whether the series converges or diverges.

EXAMPLE 5. Show that $\sum_{n=1}^{\infty} \frac{3n^2}{1-2n+7n^2}$ diverges.

Now let's end with some algebraic properties of sums, much like what we saw for sequences. Let $\sum a_n$ and $\sum b_n$ be convergent series and c a constant, then

- $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$