Lec 14: Sequences, cont'd (11.1)

Let's begin with some examples...

EXAMPLE 1. Discuss the convergence of the sequence $a_n = \dfrac{n!}{n^n}$.

EXAMPLE 2. Does the sequence $a_n = \dfrac{n^4}{n!}$ converge?

The sequence $\{r_n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases} \tag{1}$$

The book "proves" this fact (it's more of a discussion on the matter). Instead of proving it, let's just look at a couple of examples,

EXAMPLE 3. Find $\lim\limits_{n\to\infty} 2^n$

EXAMPLE 4. Find $\lim_{n\to\infty}(0.5)^n$

Theorem. If a sequence has two subsequences that converge to different limits, then the sequence diverges.

We must ask ourselves: what is a **subsequence**? A subsequence is an infinite subset of a sequence that maintains the original sequence order. While a subsequence still has an infinite number of terms, we can think of it as having less terms than the orginal sequence (even if only one term less). Given a sequence $\{x_n\}$ we can consider the subsequences: even terms, $\{x_{2n}\}$, odd terms, $\{x_{2n-1}\}$, every term but the kth term, $\{x_n\}_{n\neq k}$, etc. While the book doesn't state this theorem, the idea involved is used throughout, so it is good to have a formalism for it.

EXAMPLE 5. Find $\lim_{n\to\infty}(-1)^n$

Now let's consider sequences that are given to us **implicitly**. There are a few instances where we can take an implicit sequence and write it explicitly,

EXAMPLE 6. Write the explicit formula for $a_1 = 1,\ a_{n+1} = 5a_n$

Unfortunately we won't be able to do this most of the time. We still are interested in the convergence of implicit sequences though and so we must introduce some new terminology.

Definition. A sequence $\{a_n\}$ is called **monotonic increasing** if $a_n \leq a_{n+1}$ for all n. It is called **monotonic decreasing** if $a_n \geq a_{n+1}$ for all n. If the inequalities become strict, we say they are strictly monotonic increasing/decreasing. (Note this is slightly different then the books definition, but I introduce it since it is slightly more general).

Definition. A sequence $\{a_n\}$ is **bounded** if there exist a number M such that $|a_n| \leq M$ for all n.

EXAMPLE 7. Is the sequence $a_n = \dfrac{n}{n^2+1}$ monotonic? Is it increasing or decreasing?

EXAMPLE 8. Is the sequence $a_1 = 2,\ a_{n+1} = \sqrt{2 + a_n}$ bounded?

Proof by Induction. Proving something by induction involves two steps:

1. Prove the statement holds for the base case (usually denoted $n = 0$ or $n = 1$).
2. Prove the induction hypothesis: assume the statement holds for n and show it also holds for $n + 1$.

Theorem (Monotonic Sequence Theorem). Every bounded monotonic sequence is convergent.

EXAMPLE 9. Does the sequence $a_1 = 2,\ a_{n+1} = \dfrac{1}{2}(a_n + 6)$ converge?