In this lecture we will consider improper integrals of the second kind, those with **discontinuous integrands**. There are three cases we can consider:

1. If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

2. If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

3. If f has a discontinuity at c, where $a < c < b$, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

In the first two cases we say the integral is **convergent** if the corresponding limit exists and **divergent** if the limit does not exist (this includes infinite limits). In case (3), the integral on the left hand side exists only if **both** integrals on the right hand side exists separately (using (1) and (2) to evaluate).
The notation $t \to b^-$ and $t \to a^+$ mean we are only concerned with the limits as we approach from the left and from the right respectively.

EXAMPLE 1. Evaluate $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ if the limit exists

EXAMPLE 2. Does $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$ converge or diverge?

EXAMPLE 3. Determine whether $\displaystyle\int_{1}^{3} \frac{1}{(3-x)^{1/3}}\, dx$ converges or diverges.

EXAMPLE 4. Determine whether $\displaystyle\int_{0}^{4} \frac{1}{x-1}\, dx$ converges or diverges.

Yet again there are functions with discontinuous integrands that we wish to know the convergence or divergence of that we cannot simply integrate. The comparison theorem still holds in for these cases (just with a slight change in notation).

Theorem (comparison test for integrals). Suppose that f and g are continuous functions with discontinuities at a (b) and $f(x) \geq g(x) \geq 0$ for all $x \in (a, b]$ ($x \in [a, b)$).

(a) If $\int_a^b f(x)\,dx$ is convergent, then $\int_a^b g(x)\,dx$ is convergent.

(b) If $\int_a^b g(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is divergent.

Note: the above is not in the book, but we can convince ourselves that the results holds graphically.

It is a good idea to get a sense for what converges/diverges as then we will know which version of the comparison theorem is what we need to (try) to use. Consider the following improper integrals:

1. $\displaystyle\int_2^3 \frac{x}{\sqrt{(x-2)^3}}\,dx$

2. $\displaystyle\int_0^1 \frac{\sin^2 x}{x^2}\,dx$

3. $\displaystyle\int_0^3 \frac{dx}{x\sqrt{3-x}}$

4. $\displaystyle\int_0^1 \frac{\sin x}{x\sqrt{1-x}}\,dx$

5. $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}\,dx$

When dealing with improper integrals it is useful to go through the following checklist to solve problems successfully:

- Identify all the problem points, and whether they are type 1 or type 2.
- Split up the integral so that each integral piece only has one problem at one of its endpoints.
- Can you integrate the function without too much effort? If so integrate. Does the problem ask for a value if convergent - then you need to integrate!
- Do you think the integral converges or diverges?
- Use the appropriate version of the comparison theorem.