Lec 6: Integration of Rational Functions by Partial Fraction (7.4)

In this section we will be considering rational functions, here are a few examples:

1. $\dfrac{x+7}{x^3-4x+11}$
2. $\dfrac{x^2+x-2}{x+5}$
3. $\dfrac{4}{x^2+2x+3}$
4. $\dfrac{x^3-4}{x^2-16}$
5. $\dfrac{x^2+5}{5x^2+x-2}$

In general we can write a rational function as

$$f(x) = \frac{P(x)}{Q(x)} \tag{1}$$

where $P$ and $Q$ are polynomials. We say a rational function is **proper** if $\deg(P) < \deg(Q)$ and a rational function is **improper** if $\deg(P) \geq \deg(Q)$. Above examples (1) and (3) are proper, while (2), (4), and (5) are improper.

Before we get into the formalism of partial fractions we first must note that *we have to long divide all improper rational functions before beginning.* Partial fractions only works on proper rational functions.

**EXAMPLE 1.** Evaluate $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

Note that more often than not long dividing will not be sufficient to solve the integral and partial fractions will be necessary. Depending on the form of $Q(x)$ we need to follow slightly different procedures for partial fractions. Let us start with the simplest case:

**CASE 1:** $Q(x)$ **is the product of distinct linear factors.** This means we can write,

$$Q(x) = (a_1 x + b_1)(a_2 x - b_2) \cdots (a_n x - b_n) \tag{2}$$

where $b_1/a_1 \neq b_2/a_2 \neq \cdots \neq a_n/b_n$. In this case we can write the given proper rational function as

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_n}{a_n x + b_n} \tag{3}$$

and solve for the $A_i$'s by solving the **system of** $n$ **linear equations**.

**EXAMPLE 2.** Evaluate $\displaystyle\int \frac{x-2}{x^2+2x-3}\,dx$

**EXAMPLE 3.** Evaluate $\displaystyle\int \frac{x-4}{x^2-5x+6}\,dx$

**CASE 2: $Q(x)$ is the product of linear factors, some of which are repeated.** This means we can write,

$$Q(x) = (a_1x + b_1)^{r_1}(a_2x + b_2)^{r_2} \cdots (a_nx + b_n)^{r_n} \tag{4}$$

and the partial fraction decomposition takes the form,

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_{r_1}r}{(a_1x + b_1)^{r_1}} + \cdots + \frac{C_1}{a_nx + b_n} + \cdots + \frac{C_{r_n}}{(a_nx + b_n)^{r_n}} \tag{5}$$

Let's practice decomposing some of these partial fractions before working through the integration.

**EXAMPLE 4.** What is the decomposition of $\dfrac{x+1}{x^3(x+2)}$

**EXAMPLE 5.** What is the decomposition of $\dfrac{x^3+x+4}{(x+1)^2(x+3)^2}$

**EXAMPLE 6.** Evaluate $\int \frac{x^2 + x + 1}{(x+1)^2(x+2)}\,dx$