Second order linear equations.

Today we will consider differential equations of the form,

$$ay'' + by' + cy = 0 \quad (*)$$

which is a <u>homogeneous</u>, <u>second order</u> differential equation with <u>constant coefficients</u>.

has y''

RHS $= 0$

$a, b, c \in \mathbb{R}$
<u>NOT</u> functions of x.

Let's consider an example,

$$y'' - y' - 2y = 0 \quad (**)$$

Let's show that $y_1 = e^{-x}$ is a solution of the above.

$y_1 = +e^{-x}$

$y_1' = -e^{-x}$

$y_1'' = +e^{-x}$

$$\begin{aligned} y_1'' - y_1' - 2y_1 &= e^{-x} - (-e^{-x}) - 2(e^{-x}) \\ &= e^{-x} + e^{-x} - 2e^{-x} \\ &= (1 + 1 - 2)e^{-x} = 0. \end{aligned}$$

so LHS = RHS of $(**)$

Now let's show $y_2 = e^{2x}$ is also a solution of $(**)$,

$$y_2 = e^{2x}$$
$$y_2' = 2e^{2x}$$
$$y_2'' = 4e^{2x}$$

$$y_2'' - y_2' - 2y_2 = 4e^{2x} - 2e^{2x} - 2(e^{2x})$$
$$= (4-2-2)e^{2x} = 0$$

Now let's show $y = ae^{-x} + be^{2x}$ is also a solution of $(**)$,

a linear combination of y_1 and y_2.

$$y = ae^{-x} + be^{2x}$$
$$y' = -ae^{-x} + 2be^{2x}$$
$$y'' = ae^{-x} + 4be^{2x}$$

$$y'' - y' - 2y = (ae^{-x} + 4be^{2x}) - (-ae^{-x} + 2be^{2x}) - 2(ae^{-x} + be^{2x})$$
$$= ae^{-x} + 4be^{2x} + ae^{-x} - 2be^{-2x} - 2ae^{-x} - 2be^{2x}$$
$$= (a + a - 2a)e^{-x} + (4b - 2b - 2b)e^{2x}$$
$$= 0 + 0 = 0.$$

We can generalize this idea. Let's consider the equation $(*)$ from earlier and suppose y_1 and y_2 are solutions of $(*)$.

Let's prove $y = c_1y_1 + c_2y_2$ is also a solution of (*).

$$y = c_1y_1 + c_2y_2$$

$$y' = c_1y_1' + c_2y_2'$$

$$y'' = c_1y_1'' + c_2y_2''$$

$$ay'' + by' + cy = a(c_1y_1'' + c_2y_2'') + b(c_1y_1' + c_2y_2') + c(c_1y_1 + c_2y_2)$$

$$= c_1\underbrace{(ay_1'' + by_1' + cy_1)}_{\substack{=0 \text{ bc } y_1 \text{ is a} \\ \text{solution to } (*)}} + c_2\underbrace{(ay_2'' + by_2' + cy_2)}_{\substack{=0 \text{ bc } y_2 \text{ is a} \\ \text{solution to } (*).}}$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

<u>Thm.</u> if y_1 & y_2 are solutions to (*) then the function $y = c_1y_1 + c_2y_2$ is also a solution. Moreover, if y_1 and y_2 are <u>linearly independent</u> then the above is the <u>most general solution</u> of (*).

0 for all values of x.

$c_1y_1 + c_2y_2 \not\equiv 0$ for some c_1 & c_2

any solution of (*) can be obtained by specifying c_1 & c_2.

So the <u>general solution</u> to $(**)$ is $y = c_1 e^{-x} + c_2 e^{2x}$.

<u>Defn</u>. Given a homogeneous, second order differential equation with constant coefficients,

$$ay'' + by' + cy = 0 \quad (*)$$

we can write the <u>characteristic equation</u>,

$$ar^2 + br + c = 0 \quad (***)$$

Solutions of $(***)$ determine solutions of $(*)$.

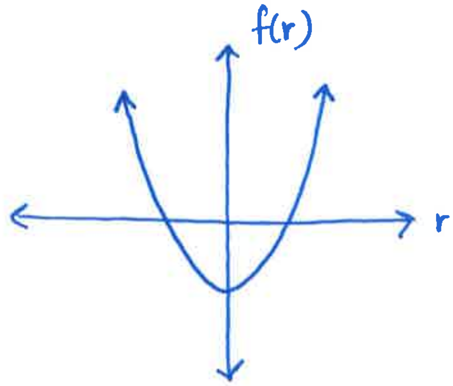
Solutions of $(***)$ $\Longleftrightarrow$ roots of a quadratic polynomial

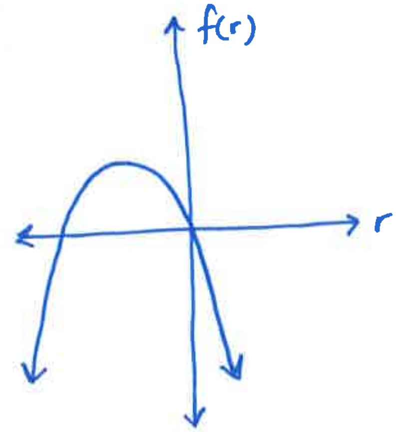
<u>Case I</u>. Two distinct real roots.

<u>Case II</u>. One real repeated root.

<u>Case III</u>. Two complex conj. roots

Case I. Two distinct real roots.





Where we graph $f(r) = ar^2 + br + c$.

The general solution to $(*)$ is $\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$ where r_1 and r_2 are the two distinct roots satisfying $ar^2 + br + c = 0$.

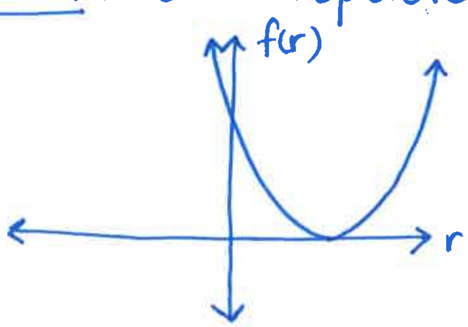
Ex1. $y'' - y = 0$

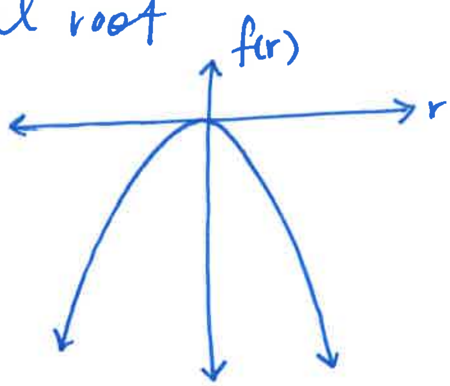
The characteristic equation is,

$$r^2 - 1 = 0 \Rightarrow \underset{r=-1}{(r+1)}\,\underset{r=1}{(r-1)} = 0$$

So the general solution is $\boxed{y = c_1 e^{-x} + c_2 e^{x}}$

Case II. One repeated real root





If the characteristic equation (***) only has one real root, r, then the general solution to (*) is,

$$\boxed{y = c_1 e^{rx} + c_2 x e^{rx}}.$$

<u>Ex 2.</u> $y'' - 6y' + 9y = 0$

The characteristic equation is,

$$r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0$$

$$\downarrow$$

$$r = 3$$ only one!

The general solution is,

$$\boxed{y = c_1 e^{3x} + c_2 x e^{3x}}$$ ⟵ let's check this solution by plugging it back in.

$$y' = 3c_1 e^{3x} + 3c_2 x e^{3x} + c_2 e^{3x} = (3c_1 + c_2) e^{3x} + 3c_2 x e^{3x}$$

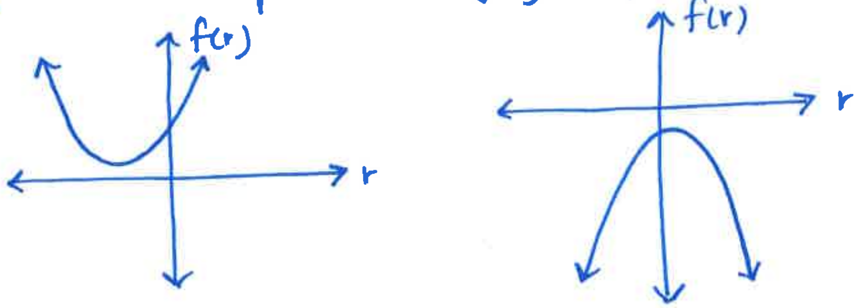
$$y'' = 3(3c_1 + c_2) e^{3x} + 9c_2 x e^{3x} + 3c_2 e^{3x}$$
$$= (9c_1 + 6c_2) e^{3x} + 9c_2 x e^{3x}$$

$$y'' - 6y' + 9y = (9c_1 + 6c_2) e^{3x} + 9c_2 x e^{3x} - 6\left[(3c_1 + c_2) e^{3x} + 3c_2 x e^{3x}\right]$$
$$+ 9\left[c_1 e^{3x} + c_2 x e^{3x}\right]$$

$$= (9c_1 + 6c_2 - 18c_1 - 6c_2 + 9c_1)e^{3x} + (9c_2 - 18c_2 + 9c_2)xe^{3x}$$

$$= 0 \cdot e^{3x} + 0xe^{3x} = 0.$$

Case III. two complex (conjugate) roots.



Suppose r_1 & r_2 are the complex roots of the characteristic equation (***), then they have the form,

$$\underbrace{r_1 = \alpha + \beta i \qquad r_2 = \alpha - \beta i}_{\text{conjugates}}, \quad \alpha, \beta \in \mathbb{R}$$

And the general solution to (*) is,

$$\boxed{y = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)}$$

Ex 3. $y'' - 6y' + 13y = 0$

The characteristic equation is,

$r^2 - 6r + 13 = 0$

The roots are,

$$r = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i \Rightarrow \begin{array}{l} \alpha = 3 \\ \beta = 2 \end{array}$$

So the general solution is,

$$\boxed{y = e^{3x}\left(c_1 \cos(2x) + c_2 \sin(2x)\right)}$$