Separable Equations (9.3)

<u>Defn</u>. A <u>separable equation</u> is a first order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y.

$$\boxed{\frac{dy}{dx} = \underbrace{f(x)}_{\text{function of just } x}\ \underbrace{g(y)}_{\text{function of just } y}}$$

Thus we can separate the x's and y's on either side of the equality,

$$\frac{dy}{g(y)} = f(x)\,dx$$

Now we can integrate both sides,

$$\boxed{\underbrace{\int \frac{dy}{g(y)}}_{\text{integrate wrt the variable } y} = \underbrace{\int f(x)\,dx}_{\text{integrate wrt the variable } x.}}$$

The prior formula defines y implicitly in terms of x. If possible, we should solve for y.

Ex 1. Solve $\frac{dy}{dx} = y$. (*) exponential growth!

$$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{1}{y}$$

$$dx \cdot \frac{dy}{y} \frac{1}{dx} = 1 \cdot dx$$

$$\frac{dy}{y} = dx$$

$$\int \frac{dy}{y} = \int dx$$

C is important here!

$$\log|y| = x + C$$

$$e^{\log|y|} = e^{x+C}$$

this is always positive

$$|y| = e^{x} \cdot e^{C}$$

this can be any real #, positive or negative

$$y = \pm e^{C} e^{x}$$

renamed the ~~variable~~ constant

$$\boxed{y = Ce^{x}}$$

Ex2. Solve $\frac{dy}{dx} = \frac{x^2}{y^2}$

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\int y^2\,dy = \int x^2\,dx$$

$$\frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

$$y^3 = x^3 + C$$

$$\boxed{y = (x^3 + C)^{1/3}}$$

Ex 1&2 solve for the family of solutions. Let's specify an initial condition to solve for a particular solution.

Ex3. Solve, $\frac{dy}{dx} = \frac{\sin x}{y}$, $y(0) = 2$

$$\frac{dy}{dx} = \frac{\sin x}{y}$$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{1}{2}y^2 = -\cos x + C$$

redefine C here

both sign are important $y = \pm\sqrt{-2\cos x + C}$

Now use the initial conditions,

since 2 is positive we choose the positive branch

$$2 = +\sqrt{-2\cos(0) + C}$$ solve for C

$$2 = \sqrt{-2(1) + C}$$

$$4 = -2 + C \Rightarrow C = 6$$

$$\boxed{y = \sqrt{-2\cos x + 6}}$$

Ex 4. Solve $\frac{dy}{dx} = x^2 + x^2y^2$

$$\frac{dy}{dx} = x^2(1+y^2)$$

$$\int \frac{dy}{1+y^2} = \int x^2\,dx$$

$$\arctan y = \frac{1}{3}x^3 + C$$

$$y = \tan\left(\tfrac{1}{3}x^3 + C\right)$$

Ex5. Solve $\dfrac{dy}{dx} = -\dfrac{x}{y-3}$, $y(0) = 1$

$$\frac{dy}{dx} = \frac{-x}{y-3}$$

$$\int (y-3)\,dy = \int -x\,dx$$

$$\left(\tfrac{1}{2}y^2 - 3y = -\tfrac{1}{2}x^2 + C\right) \times 2$$

$$y^2 - 6y = -x^2 + C$$

Solve for y by using the quadradratic equation,

$$y^2 - 6y + \overbrace{(x^2 + C)}^{\text{"constant term"}} = 0$$

$$y = 3 \pm \sqrt{9 - x^2 + 2C}$$ combine these

$$y = 3 \pm \sqrt{C - x^2}$$

Now use initial condition,

$$1 = 3 \pm \sqrt{C} \Rightarrow \pm\sqrt{C} = -2 \Rightarrow C = 4.$$