Lecture 30: notes

Tuesday, August 6, 2019 1:30 PM

We've seen that not all first order differential equations are separable, but luckily there is another class we can consider.

Def'n. A first order differential equation is called linear if we can write it in the form,

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)}$$

continuous functions of x.

For the following answer the questions, (1) is it linear, and (2) if so what are $p(x)$ and $q(x)$.

(A) $\frac{dy}{dx} - 3x = 0 \Rightarrow \frac{dy}{dx} = 3x$

(1) Yes (2) $p(x) = 0$ $q(x) = 3x$

(1) Yes (2) $p(x) = 0$, $q(x) = 3x$

(B) $\dfrac{dy}{dx} + xy^{2} = x$

(1) Not linear

(C) $\dfrac{dy}{dx} = \sin(x) \cdot y + \cos(x)$

$$\frac{dy}{dx} - \sin(x) \cdot y = \cos(x)$$

(1) Yes (2) $p(x) = -\sin(x)$, $q(x) = \cos(x)$

(D) $x^2 \dfrac{dy}{dx} = y \Rightarrow \dfrac{dy}{dx} - \dfrac{y}{x^2} = 0$

(1) Yes (2) $p(x) = -\dfrac{1}{x^2}$, $q(x) = 0$

Once we have a solution in the appropriate form we can learn how to solve,

$\mu(x)$ – integrating factor

We will define $\mu(x)$ explicitly later. For now we define it as the function that satisfies

$$y' + py = q$$

multiply through by μ

$$\mu y' + p\mu y = \mu q$$

$$\mu y' + p\mu y = \mu q$$

$$\mu y' + \mu' y = \mu q$$

Notice the LHS is the product rule of diff.

$$(\mu y)' = \mu q$$

integrate both sides

$$\int (\mu y)' \, dx = \int \mu q \, dx$$

$$\mu y = \int \mu q \, dx$$

$$\boxed{y = \frac{\int \mu q \, dx}{\mu}}$$

make sure to add the constant from integration.

So now the question becomes what is $\mu(x)$?

$$\mu' = p\mu$$

$$\frac{\mu}{\mu} = p$$

$$(\log \mu)' = p$$

$$\log \mu = \int p(x)\,dx$$

not necessary to add constant of integration

$$\boxed{\mu(x) = e^{\int p(x)\,dx}}$$

<u>Ex1.</u> $\dfrac{dy}{dx} - \dfrac{y}{x} = 1$

From the above we see that $p(x) = -\frac{1}{x}$ and $q(x) = 1$.

$$\mu(x) = e^{\int -\frac{1}{x}dx}$$

$$= e^{-\log x} = e^{\log x^{-1}} \Rightarrow \mu(x) = \frac{1}{x}$$

$$y = \frac{1}{\mu(x)} \int q(x)\mu(x)\,dx$$

$$= \frac{1}{\frac{1}{x}} \int 1 \cdot \frac{1}{x}\,dx = x \cdot (\log x + C)$$

$$= x \log x + Cx$$

Let's check the solution we get:

$$y' = \log x + 1 + C$$

LHS: $\log x + 1 + C - \dfrac{x\log x + Cx}{x}$

$$= \cancel{\log x} + 1 \cancel{+ C} - \cancel{\log x} \cancel{- C} = 1$$

RHS: 1

The process is as follows,

(1) Put the equation in the form
$\dfrac{dy}{dx} + p(x)y = q(x)$

(2) Compute $\mu(x)$.

(3) Compute $y(x)$

(4) Check (optional).

<u>Ex2.</u> $\dfrac{dy}{dx} + 2xy = -2x^3$

(1) ✓

(2) $\mu(x) = e^{\int 2x\,dx}$

$$= e^{x^2}$$

(3) $y(x) = \frac{1}{e^{x^2}} \int -2x^3 e^{x^2}\, dx$

$$= \frac{-1}{e^{x^2}} \underbrace{\int 2x^3 e^{x^2}\, dx}$$

$u = x^2$

$du = 2x\, dx$

$$= \int u e^u\, du$$

$$= ue^u - e^u = x^2 e^{x^2} - e^{x^2}$$

$$y(x) = -x^2 + 1 + \frac{c}{e^{x^2}}$$

(4) exercise.

Ex3. $y' + y(\cos x) = \cos x$, $y(0) = 2$.

(1) ✓

(2) $\mu(x) = e^{\int \cos x\, dx}$

$$= e^{\sin x}$$

(3) $y(x) = \frac{1}{e^{\sin x}} \cdot \int \cos x \cdot e^{\sin x} \, dx$

$$= \frac{1}{e^{\sin x}} \cdot \left(e^{\sin x} + C \right)$$

$$y(x) = 1 + Ce^{-\sin x}$$

(3') *apply initial condition*

$$2 = 1 + Ce^{-\sin 0}$$
$$2 = 1 + C$$
$$1 = C$$

$$y(x) = 1 + e^{-\sin x}$$

(4) exercise.

Practice problems:

(1) $y' = x + 5y$

(2) $y' - 3y = e^x$

(3) $x^2y' + 2xy = \cos x \quad (x>0)$

(4) $y' + y = x + e^x, \quad y(0) = 0$

(5) $(1+x^2)y' + 2xy = 3\sqrt{x}, \quad y(0) = 2.$