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# Differential equations.

Today we will answer the question: what is a differential equation? And motivate the use of differential equations for solving real word problems (i.e. modelling phenomena).

Def'n. A differential equation is an equation containing derivatives of a function. The following are examples of differential equations,

$$y' = x^2 + y$$

$$f'(x) = f(x) + 2x - 3$$

$$P' = kP$$

The above are all examples of first order differential equations, because they are an equation for the first derivative of the function. Notice that the equation also includes the input of the function and the function itself.

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# Differential equations as population models.

Consider the following example:

A scientist is studying a group of bunnies and notices that the *growth of the population is directly proportional to the size of the population*. Let's define some variables,

$P$  = the population of the group of bunnies.

$\frac{dP}{dt} = P'$  = the growth of the population.

$k$  = some proportionality constant.

Thus we get,

$$P' = kP$$

Let's set  $k = 1$  and create a directional field for the population model.

## Directional field

A directional field uses a small arrow to indicate the slope at a given point  $(x,y)$  -- or in this example  $(P,t)$ . Let's compute a few values,

$$(0, 0): P' = P = 0$$

$$(1, 0): P' = 0$$

$$(x, 0): P' = 0$$

$$(1, 2): P' = 2$$

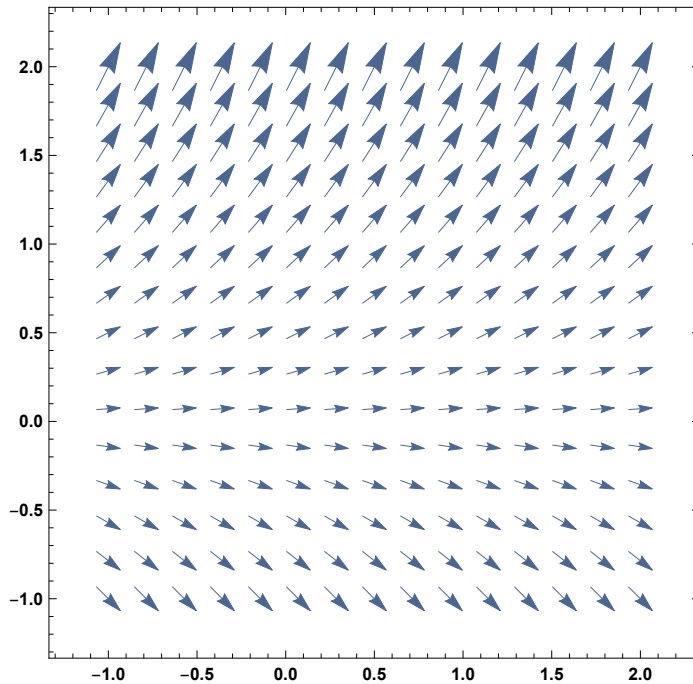
$$(0, 2): P' = 2$$

$$(x, 2): P' = 2$$

$(x, y): P' = y$

Now we can plot these little vectors on a graph...

```
popvec = VectorPlot[{1, y}, {x, -1, 2}, {y, -1, 2}]
```



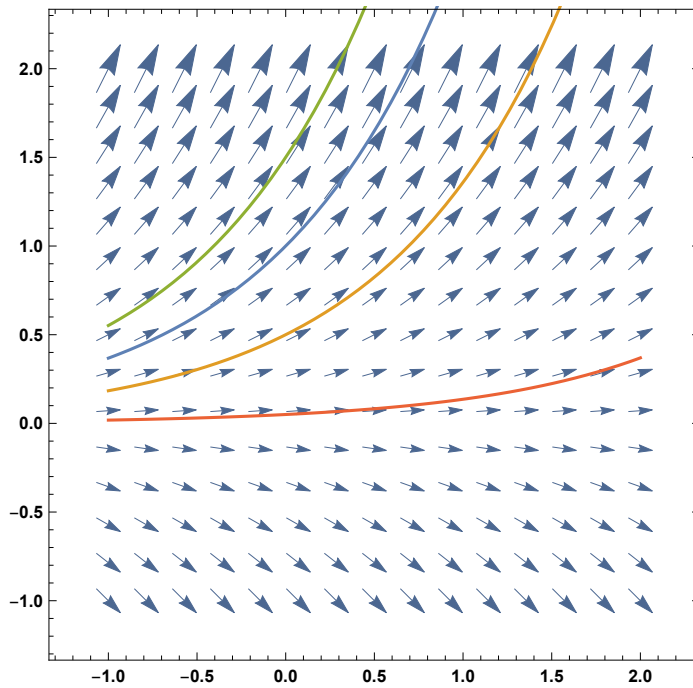
Above the bigger arrows indicate the the slope is steeper (that is, greater in magnitude).

We can use these "guide lines" to draw trajectories of the function  $P(t)$  by specifying different initial conditions  $P(0) = p_0$ .

```
poptraj = Plot[{E^x, 0.5 E^x, 1.5 E^x, 0.05 E^x}, {x, -1, 2}];
```

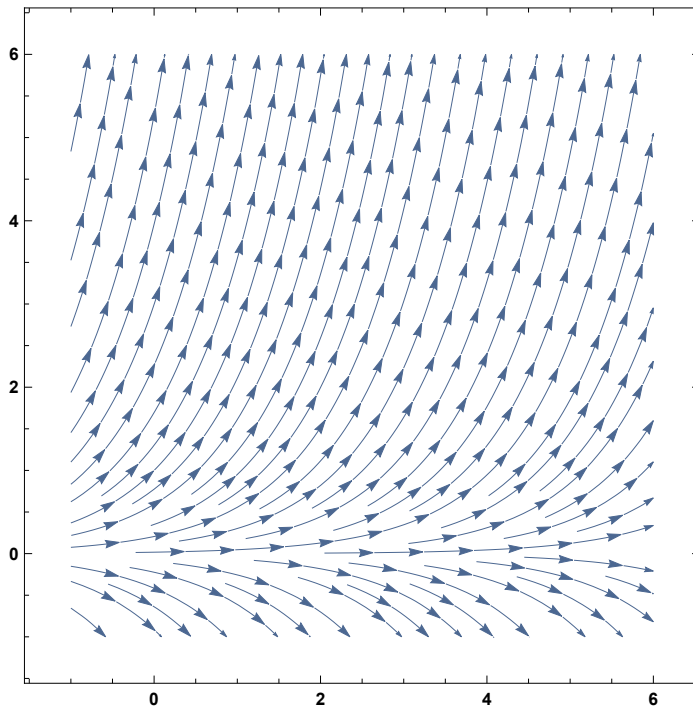
In the above I've set  $P(0) = 1, 0.5, 1.5,$  and  $0.05$ .

```
Show[popvec, poptraj]
```



Mathematica combines the above two things into a nice function called `StreamPlot`, so let's check out what that looks like.

```
StreamPlot[{1, y}, {x, -1, 6}, {y, -1, 6}]
```



**Solution to the population model.**

Let's show that the function  $P(t) = p_0 e^t$  satisfies the above model. Moreover, let's show that  $p_0$  is the initial population.

$$\begin{aligned} P(t) &= p_0 e^t \\ P'(t) &= p_0 e^t \quad \text{by taking the derivative} \\ P'(t) &= P(t) \end{aligned}$$

$$\begin{aligned} P(0) &= p_0 e^0 \\ P(0) &= p_0 \cdot 1 = p_0 \end{aligned}$$

Because of the solution we call this the exponential growth model.

We see that we have the family of solutions to the model. Note that most first order differential equations have a family of solutions. We can pick out one solution by setting the initial condition (sometimes called the initial value or boundary condition).

## Another population model.

The bunny model has its shortcomings as a population model. Why? Because in the model the bunnies can grow on forever, but in the real world there are often limiting factors to the size of a population.

Often time resources such as space, food, or competition prevent populations from getting too big. So let's try to express this mathematically...

When the population is small let's say  $P' \sim kP$  (in other words, when the population is small growth looks exponential). When the population nears some maximum population  $M$  it slows down and virtually stops growing. So we write,

$$P' = kP\left(1 - \frac{P}{M}\right)$$

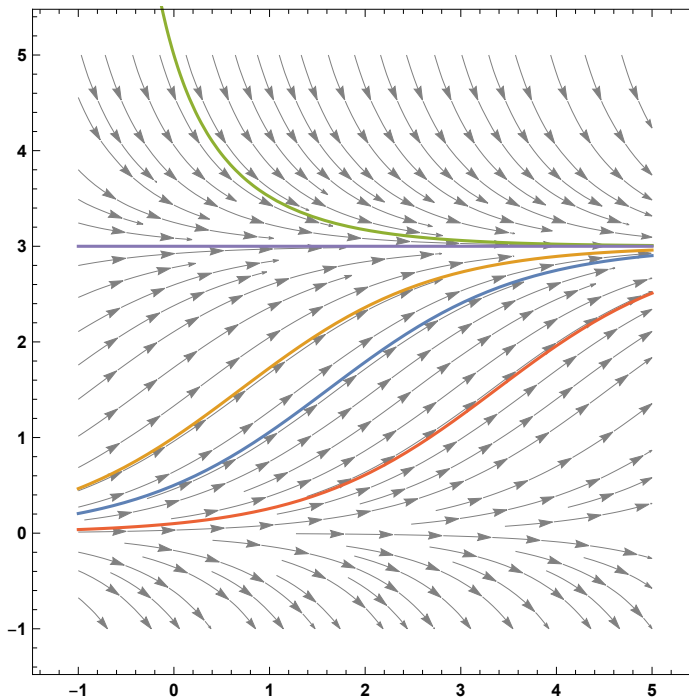
We can see in the above that:

$$\begin{aligned} P' &= 0 \text{ when } P = 0 \text{ and } P = M \\ P' &> 0 \text{ when } 0 < P < M \\ P' &< 0 \text{ when } P > M \text{ ( \& } P < 0 \text{)}. \end{aligned}$$

Let's graph the StreamPlot, along with different solutions (set  $k = 1$ ,  $m = 3$ ).

```
pop2stream = StreamPlot[{1, y (1 - y/3)}, {x, -1, 5}, {y, -1, 5}, StreamStyle -> Gray];
DSolve[{p'[x] == p[x] (1 - p[x]/3), p[0] == a}, p, x];
pop2traj = Plot[
  Evaluate[p[x] /. % /. {{a -> 1/2}, {a -> 1}, {a -> 5}, {a -> 0.1}, {a -> 3}}], {x, -1, 5}];
```

Show[pop2stream, pop2traj]



Notice that we haven't actually solved for the function  $P(t)$ , yet we have a good idea of the qualitative characteristics of the functions (depending on its initial condition of course).

Here the family of solutions don't all look the same. Depending on the initial condition the function could look completely different (i.e.  $P(0) = 1$  versus  $P(0) = 5$ ). Any constant solutions (i.e.  $P(t) = 3$  and  $P(t) = 0$ ) are called equilibrium solutions. Because the system being modeled isn't changing -- it's at equilibrium.

## A general first order differential equation.

We can extend the analysis above to any function we desire. Let's consider,

$$y' = x(y - 1)$$

$$(0,0): y' = 0$$

$$(0,y): y' = 0$$

$$(x,1): y' = 0$$

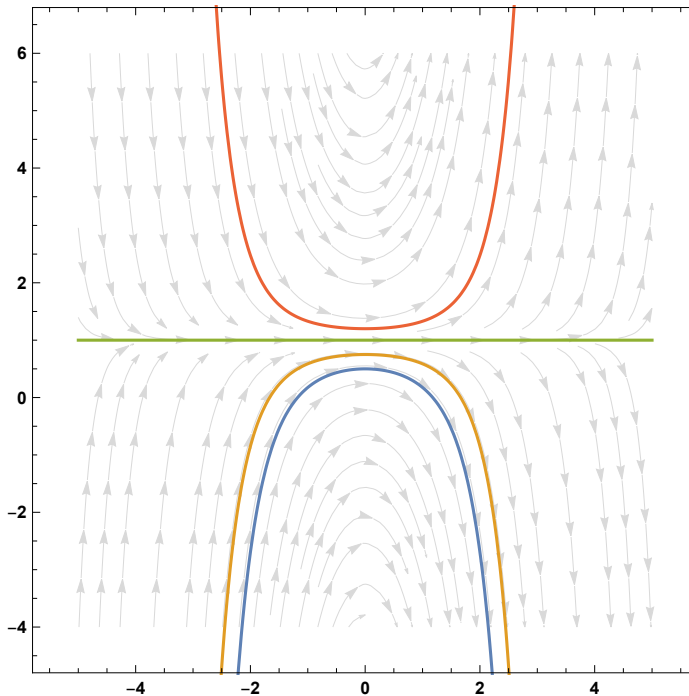
$$(2,2): y' = 2$$

$$(1,2): y' = 1$$

```

randstream =
  StreamPlot[{1, x (y - 1)}, {x, -5, 5}, {y, -4, 6}, StreamStyle -> LightGray];
DSolve[{y' [x] == x (y[x] - 1), y[0] == a}, y, x];
randtraj =
  Plot[Evaluate[y[x] /. % /. {{a -> 1/2}, {a -> 0.75}, {a -> 1}, {a -> 1.2}}], {x, -5, 5}];
Show[randstream, randtraj]

```



So now the question becomes, what is the precise solution to the above differential equations? In some cases it's easy, in some it's hard, and in some it's impossible. We will dive into how to solve these in the next few days.

## Another one.

Let's show that  $y(x) = \frac{1+ce^x}{1-ce^x}$  is the solution to the differential equation  $y' = \frac{1}{2}(y^2 - 1)$ .

Differentiate the function,

$$y'(x) = \frac{(1 - ce^x) ce^x + (1 + ce^x) ce^x}{(1 - ce^x)^2} = \frac{2 ce^x}{(1 - ce^x)^2}$$

Now compute the right hand side of the differential equation and see if it matches with the above,

$$\frac{1}{2}(y^2 - 1) = \frac{1}{2} \left( \left( \frac{1+ce^x}{1-ce^x} \right)^2 - 1 \right) = \frac{1}{2} \left( \frac{4 ce^x}{(1-ce^x)^2} \right) = \frac{2 ce^x}{(1-ce^x)^2}$$

And so the family of equations,  $y(x)$ , are a solution to the differential equation. Let's plot a few for fun.

```
Plot[  
  {Evaluate[ $(1 + c E^x) / (1 - c E^x)$ ] /. {{c -> 1/2}, {c -> 0.75}, {c -> 3}, {c -> -3}},  
  1, -1}, {x, -5, 5}]
```

