Lecture 26: notes

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Today we would like to develop more tools for computing Taylor series.

<u>Ex 1.</u> Compute the Taylor series of the function, $f(x) = e^x + \frac{1}{(1-x)^2}$ about $x=0$.

Consider the Taylor series of each piece,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

take the derivative

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

shift index

$$= \sum_{n=0}^{\infty} (n+1) x^n$$

Now let's add the two,

$$e^x + \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} (n+1) x^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!} + n + 1\right) x^n$$

either is okay.

$$= \sum_{n=0}^{\infty} \frac{1+(n+1)!}{n!} x^n$$

When adding together Taylor series add the coefficients on the same powers of x.

Ex2. Compute the Taylor series about $x=0$ of the function $f(x)=\dfrac{2-x^2}{1+x}$.

$$f(x)=\frac{2-x^2}{1+x}$$

find the Taylor series of these.

$$=\frac{2}{1+x}-\frac{x^2}{1+x}$$

$$=2\cdot\sum_{n=0}^{\infty}(-x)^n-x^2\cdot\sum_{n=0}^{\infty}(-x)^n$$

$$=\sum_{n=0}^{\infty}2\cdot(-1)^n x^n-\sum_{n=0}^{\infty}(-1)^n x^{n+2}$$

$$=2-2x+2x^2-2x^3+\cdots+x^2-x^3+\cdots$$

$$=2-2x+\underbrace{3x^2-3x^3+\cdots}$$

all the rest of the coefficients are 3.

$$\frac{2-x^2}{1+x}=\sum_{n=0}^{\infty}(-1)^n b_n x^n,\quad b_n=\begin{cases}2 & \text{if } n\le 1\\ 3 & \text{if } n\ge 2.\end{cases}$$

We can also multiply and divide Taylor series like we multiply and divide polynomials.

Ex3. Compute the first 4 terms of the Taylor series for $f(x)=e^x\cdot\dfrac{1}{1-x}$.

$$e^x\cdot\frac{1}{1-x}=\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\cdot\left(\sum_{n=0}^{\infty}x^n\right)$$

x^2 terms

$$=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)\cdot\left(1+x+x^2+x^3+\cdots\right)$$

constant terms $\qquad$ x^1 terms

$$= 1 + 1\cdot x + x\cdot 1 + 1\cdot\frac{x^2}{2} + x^2\cdot 1 + x\cdot x + \frac{x^3}{6}\cdot 1 + 1\cdot x^3 + x\cdot\frac{x^2}{2} + x^2\cdot x + \dots$$

$$= 1 + 2x + \left(\frac{1}{2}+2\right)x^2 + \left(\frac{1}{6}+1+\frac{1}{2}+1\right)x^3 + \dots$$

$$= 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots$$

<u>Ex4.</u> Compute the first 4 terms of the Taylor series $f(x) = e^x \cdot \frac{1}{1-x}$ (the same one!) about $x=0$.

Now let's divide the Taylor series for e^x by $1-x$,

$$\begin{array}{r l}
 & 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\dots \\
1-x \,\big) & 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\dots \\
 & -(1-x) \\ \hline
 & 2x+\frac{1}{2}x^2 \\
 & -(2x-2x^2) \\ \hline
 & \frac{5}{2}x^2+\frac{1}{6}x^3 \\
 & -\left(\frac{5}{2}x^2-\frac{5}{2}x^3\right) \\ \hline
 & \frac{8}{3}x^3
\end{array}$$

order from smaller powers of x to larger powers of x.

And just like above we get $\frac{e^x}{1-x} = 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\dots$

Note that we usually can't find a general formula for when multiplying or dividing series.

<u>Ex5.</u> Compute the first 3 terms of the Maclaurin series of $f(x) = e^x \log(1+x)$

Write a term or

$e^x = \sum_{n=0} \frac{\quad}{n!}$, $\log(1+x) = \sum_{n=1} (-1) \frac{\quad}{n}$

two more than what you need.

$$e^x \cdot \log(1+x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right) \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\right)$$

$$= \underbrace{1 \cdot x}_{x \text{ term}} + \underbrace{x \cdot x + 1 \cdot \left(-\frac{x^2}{2}\right)}_{x^2 \text{ term}} + \underbrace{1\left(\frac{x^3}{3}\right) + x\left(-\frac{x^2}{2}\right) + x\left(\frac{x^2}{2}\right)}_{x^3 \text{ term}} + \dots$$

$$= x + \left(1 - \frac{1}{2}\right)x^2 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{2}\right)x^3 + \dots$$

$$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

Ex6. Compute the first 3 terms of the Maclaurin series of $f(x) = \frac{e^x}{1+x^2}$.

$$\boxed{1 + x - \frac{1}{2}x^2}$$

$$1 + 0x + x^2 \;\big)\; 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$$

$$-(1 + 0x + x^2)$$

$$x - \frac{1}{2}x^2 + \frac{1}{6}x^3$$

$$-(x + 0x^2 + x^3)$$

$$-\frac{1}{2}x^2 - \frac{5}{6}x^3 + \dots$$

order from smallest power to largest.