Lecture 25: notes

Sunday, July 28, 2019 8:30 PM

Last lecture was built around the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

function

power series or Taylor series

but there are A LOT of functions that don't come from $1/(1-x)$ but we would still like to write a power series representation.

Goal: learn how to write the Taylor series of any function, $f(x)$.

Let's assume $f(x)$ is infinitely differentiable, meaning I can take as many derivatives of $f(x)$ as I desire.

Examples. $f(x)=e^x$, $f(x)=\sin x$, $f(x)=x^2+3x$, ...

Def'n. Let $f(x)$ be an infinitely differentiable function with a power series representation centered at $x=a$. then the Taylor series of

$f(x)$ about $x=a$ is,

the nth derivative of the function at a

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad 0! = 1. \\ &= f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \dots \end{aligned}}$$

A Taylor series centered at $x=0$ is often referred to as a Maclaurin series.

Ex1. Compute the Maclaurin series of $f(x)=e^x$.

$f(0) = e^0 = 1$

$f'(x) = e^x, \quad f'(0) = 1$

$f''(x) = e^x, \quad f''(0) = 1 \quad \dots$

All derivatives are e^x. $\Rightarrow$ $f^{(n)}(0) = 1$.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Ex2. Compute the Taylor series of the function $f(x) = \frac{1}{x}$ about $x=1$.

$f(x) = 1/x \qquad f(1) = 1$

$f'(x) = -1/x^2 \qquad f'(1) = -1$

$f''(x) = 2 \cdot 1/x^3 \qquad f''(1) = 2 \cdot 1 = 2!$

$f'''(x) = -3 \cdot 2 \cdot 1/x^4 \qquad f'''(1) = -3 \cdot 2 \cdot 1 = -3!$

get a pattern out of this.

$f^{(n)}(x) = (-1)^n n \cdot (n-1) \cdots 2 \cdot 1/x^{n+1} \qquad f^{(n)}(1) = (-1)^n n!$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{n!}}{\cancel{n!}} (x-1)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

Of course, using the formula isn't always the best way to find a Taylor series.

Ex3. Compute the Taylor series of the function $f(x) = e^{x^2}$.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

The above was MUCH easier than computing all those derivatives.

Ex4. What is the radius of convergence of the Maclaurin series of $f(x)=e^x$.

The Taylor series of $f(x)=e^x$ is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1} \cdot n!}{(n+1)! \cdot x^n} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot |x| = 0$$

So $R=\infty$, in other words the Taylor series represents the function everywhere.

Ex5. Compute the value of the series, $\sum_{n=0}^{\infty} \frac{1}{n!}$.

Here we will use the Taylor series for e^x,

$$e^x = \sum \frac{x^n}{n!} \Rightarrow e^1 = \sum \frac{(1)^n}{n!}$$

$$\Rightarrow e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Now we have a new method for computing the sum of a series.