Lecture 24: notes

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Yesterday we spent a lot of time on the power series,

$$f(x) = \sum_{n=0}^{\infty} x^n$$

we determined that the power series converges for all x such that $|x| < 1$.

We can compute,

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

This is only true for $|x| < 1$.

Uses geometric sum formula

(*) Graph of function & partial sums.

<u>Ex1</u>. Express the function $f(x) = \frac{1}{1-3x}$ as a power series.

We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

replace x with $3x$

$$\frac{1}{\quad} = \sum^{\infty} (3x)^n$$

$$1-3x \qquad \sum_{n=0} (3x)$$

$$= \sum_{n=0}^{\infty} 3^n x^n .$$

Ex2. Express the function $f(x) = \dfrac{x^2}{1-x}$ as a power series,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$x^2 \cdot \frac{1}{1-x} = x^2 \cdot \sum_{n=0}^{\infty} x^n$$

add exponents on like bases.

$$= \sum_{n=0}^{\infty} x^2 \cdot x^n = \sum_{n=0}^{\infty} x^{n+2}$$

Exercise. Express the function $f(x) = \dfrac{1}{1-x^2}$ as a power series.

Answer. $f(x) = \sum_{n=0}^{\infty} x^{2n}$.

Ex3. Express the function $f(x) = \dfrac{x}{1+x^3}$ as a power series,

We start with the only function we

know, $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

$$x \cdot \frac{1}{1+x^3} = x \cdot \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

$$= \sum_{n=0}^{\infty} (-1)^n x \cdot x^{3n}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n+1}$$

Exercise. Express the function $f(x) = \frac{x^2}{1+2x}$ as a power series.

Answer. $\sum_{n=0}^{\infty} (-1)^n \cdot 2^n x^{n+2}$.

Let's go back to the function $f(x) = \frac{1}{1-x}$ and take it's derivative,

$$f'(x) = \frac{1}{(1-x)^2}$$

Can I write the power series for $f'(x)$

since I know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$?

YES.

Here's a theorem...

<u>Thm.</u> If the power series $\sum c_n (x-a)^n$ has a radius of convergence $R > 0$, then the function $f(x)$ defined by,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$
$$= c_0 + c_1 x + c_2 x^2 + \dots$$

is differentiable on the interval $(a-R, a+R)$ and,

(i) $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n-1}$

Both power series have radius of

convergence R.

The above is called term by term differentiation and integration, because we differentiate/integrate each term of the power series.

Ex4. Compute the power series of the function $f(x) = \frac{1}{(1-x)^2}$.

derivative

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n)$$

take the derivative of each term.

$$= \sum_{n=1}^{\infty} n \cdot x^{n-1}$$

Exercise. Let $f(x) = \frac{x}{1-x}$. Determine the power series of $f'(x)$.

Answer. $f(x) = \sum_{n=0}^{\infty} x^{n+1}$

$$f'(x) = \sum_{n=0}^{\infty} (n+1) x^n$$

Ex5. Compute the power series of the function $f(x) = \log(1+x)$.

We have to make the observation,

$$f'(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

doesn't depend on x.

$$\int \frac{1}{1+x}\,dx = \log(1+x) = \sum_{n=0}^{\infty} (-1)^n \int x^n\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

need to add a constant term

What is this constant C? It's the value of the function at the center of the power series,

$$C = \log(1+0) = 0$$

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}.$$

Exercise. Compute the power series of
$f(x) = \log(1+x^2)$

Answer. $\log(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2n+2}$